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Consensus of Multi-Agent Networks With Aperiodic Sampled Communication Via Impulsive Algorithms Using Position-Only Measurements

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***Abstract*—In this technical note, an impulsive consensus algorithm is proposed for second-order continuous-time multi-agent networks with switching topology. The communication among agents occurs at sampling instants based on position only measurements. By using the property of stochastic matrices and algebraic graph theory, some sufficient conditions are obtained to ensure the consensus of the controlled multi-agent network if the communication graph has a spanning tree jointly. A numerical example is given to illustrate the effectiveness of the proposed algorithm.**

***Index Terms*—Aperiodic sampled information, consensus, impulsive algorithms, multi-agent networks.**

I. Introduction

It is well known that the consensus problem of multi-agent networks has been widely investigated due to its important applications, including coordinated control of mobile robots, synchronization of dynamical networks, distributed Kalman filtering in sensor networks, load balancing in parallel computers, etc [1]–[4]. Many results have been reported for multi-agent networks with different special features, such as time delay [5], switching topology [6], asynchronous algorithms [4], [7], nonlinear algorithms [8], [9], quantized data [4], noisy communication channel [10], second-order model [11], [12], optimal consensus [13], etc.

Most of the existing works on continuous-time multi-agent networks assume continuous communication among agents. However, in many real-world networks, communication among agents may occur periodically rather than continuously. Therefore, it is more practical to consider continuous-time multi-agent networks with communication at sampling instants. In [16]–[21], consensus problems were addressed for continuous-time multi-agent networks with sampled-data setting. But, all those works assume an equidistant sampling interval, and those results cannot be directly applied to systems whose length of sampling interval is time-varying or uncertain. Consequently, it is desirable to study the consensus problem of multi-agent networks with aperiodic sampling interval. In addition, the existing works often assume that each agent can obtain the information of its full states. However, in some cases, partial states may be unavailable because of technology

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limitations or communication constraints. It is more practical to realize the consensus by utilizing partial states. Hong *et al.* [14] investigated the leader-following consensus of multi-agent networks without using the leader's velocity. In [11], the consensus problem of second-order multi-agent networks in undirected networks with fixed topology was studied, where each agent can only obtain its positions relative to its neighbors. Gao *et al.* [23] considered the consensus problem of multi-agent networks with time-delay, where each agent can only obtain the measurements of its position relative to its neighbors. Yu *et al.* [24] proposed a consensus protocol using both continuous and sampled position data without using any velocity information of agents. However, all these mentioned works [11], [14], [23], [24] require continuous communication among agents, which is not desirable in some applications.

In this technical note, we investigate the consensus problem of continuous-time second-order multi-agent networks with switching topology. It is assumed that communication among agents occurs at sampling instants, and each agent can only obtain the relative positions to its neighbors and the relative position to its own state at previous sampling instant. It is also assumed that the sampling period is time-varying. Motivated by impulsive control strategy [25], [26], an impulsive consensus algorithm is proposed. Some sufficient conditions are given to ensure the consensus of the multi-agent network if the communication graph has a spanning tree jointly.

II. Preliminaries

We first present some mathematical notations to be used throughout this technical note. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N} = 1, 2, 3 \ldots$. The identity matrix of order n is denoted as I_n (or simply I if no confusion arises). $\mathbf{1}_n = (1, 1, \ldots, 1)^T$ is the column vector. $\mathbf{0}_{n\times m}$ (or simply $\mathbf{0}$ if no confusion arises) denotes the $n \times m$ matrix where all elements are equal to zero. The matrix A is non-negative, i.e., $A \geq \mathbf{0}$, if all elements of A are non-negative. For matrices $A, B \in \mathbb{R}^{N\times N}$, $A \geq B$ denotes $A - B \geq \mathbf{0}$.

Lemma 1: [15] Let $m \geq 2$ be a positive integer and $A_1, A_2, \ldots, A_m$ be non-negative $N \times N$ matrices with positive diagonal entries, then $A_1 A_2 \ldots A_m \geq \varepsilon(A_1 + A_2 + \ldots + A_m)$, where $\varepsilon > 0$ can be specified from matrices $A_i, i = 1, 2, \ldots, m$.

The non-negative matrix A is row stochastic if the sum of all elements of its row is equal to 1. The row stochastic matrix $A \in \mathbb{R}^{N\times N}$ is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} A^k = \mathbf{1}_N y^T$, where y is some $N \times 1$ column vector.

Lemma 2: [22] Let $\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_k \in \mathbb{R}^{N\times N}$ be a finite set of SIA matrices with the property that for each sequence $\mathbf{P}_{i_1}, \mathbf{P}_{i_2}, \ldots \mathbf{P}_{i_j}$ with positive length, the matrix product $\mathbf{P}_{i_1}\mathbf{P}_{i_2}\ldots\mathbf{P}_{i_j}$ is SIA. Then, for each infinite sequence $\mathbf{P}_{i_1}, \mathbf{P}_{i_2}, \ldots \mathbf{P}_{i_j}, \ldots$, there exists a column vector y such that $\lim_{j\to\infty} \mathbf{P}_{i_1}\mathbf{P}_{i_2}\ldots\mathbf{P}_{i_j} = \mathbf{1}_N y^T$.

A directed graph (digraph) will be used to model network topology among agents. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a directed graph of order N, where $\mathcal{V} = \{1, 2, \ldots N\}$ is the set of nodes, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = (a_{ij})_{N\times N}$ is the weighted adjacency matrix. The node i represents the agent i, and an edge in $\mathcal{G}$ is denoted by an ordered pair $\{j, i\}$. $\{j, i\} \in \mathcal{E}$ if and only if the agent i can directly receive information from the jth agent. Then, the set of neighbors of the node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The elements associated with the edges are positive, i.e., $j \in \mathcal{N}_i \Leftrightarrow a_{ij} > 0$, and assume that $a_{ii} = 0, i \in \mathcal{V}$. Let $\deg(i) = \sum_{j=1}^N a_{ij}$, $\mathcal{D} = diag\{\deg(1), \deg(2), \ldots, \deg(N)\}$, the Laplacian matrix of the digraph $\mathcal{G}$ is defined as $L = \mathcal{D} - \mathcal{A}$. A directed path in a digraph $\mathcal{G}$ is an ordered sequence $v_1, v_2, \ldots, v_l$ of agents such that any ordered pair of vertices appearing consecutively in the sequence is an edge of the digraph, i.e. $(v_i, v_{i+1}) \in \mathcal{V}$, for any $i = 1, 2, \ldots, l-1$. A directed tree is a digraph, where there exists an agent, called the root, such that any other agent of the digraph can be reached by one and only one path starting at the root. $\mathcal{T}_\mathcal{G} = \{\mathcal{V}_\mathcal{T}, \mathcal{E}_\mathcal{T}\}$ is a spanning tree of $\mathcal{G}$, if $\mathcal{T}_\mathcal{G}$ is a directed tree and $\mathcal{V}_\mathcal{T} = \mathcal{V}$.

Lemma 3: [1] Let L be Laplacian matrix of the digraph $\mathcal{G}$. Zero is a simple eigenvalue of L, and all the other eigenvalues have positive real parts if and only if $\mathcal{G}$ contains a spanning tree.

The union of the digraphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_k$ with the same node set $\mathcal{V}$ is a directed graph with the node set $\mathcal{V}$ and the edge set as the union of the edge sets of the directed graphs in the collection.

Given a matrix $\mathbf{P} = (p_{ij}) \in \mathbb{R}^{N\times N}$, the digraph of $\mathbf{P}$, denoted by $\mathcal{G}(\mathbf{P})$, is the digraph with the node set $\mathcal{V} = \{1, 2, \ldots, N\}$ such that there is an edge in $\mathcal{G}(\mathbf{P})$ from j to i if and only if $p_{ij} \neq 0$.

Lemma 4: [18] The stochastic matrix A has algebraic multiplicity equal to one for its eigenvalue $\lambda = 1$ if and only if the digraph $\mathcal{G}(A)$ has a spanning tree.

Lemma 5: [18] Suppose that $\mathbf{P} \in \mathbb{R}^{N\times N}$ is a row stochastic matrix with positive diagonal elements. If the digraph $\mathcal{G}(\mathbf{P})$ has a spanning tree, then $\mathbf{P}$ is SIA.

Consider a multi-agent network consisting of N identical agents indexed by $1, 2, \ldots, N$, which is described by

$$\dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t) \tag{1}$$

where $i = 1, 2, \ldots, N$, $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$ are the position and velocity states of the agent i, respectively. $u_i(t) \in \mathrm{R}^n$ is a control input for the agent i.

Definition 1: Consensus in the multi-agent network (1) is said to be achieved, if for any initial state, $\lim_{t\to\infty} x_i(t) = \zeta$ and $\lim_{t\to\infty} v_i(t) = 0$, where $i \in \mathcal{V}$ and $\zeta \in \mathbb{R}$ is a constant.

In this technical note, we consider communication among agents occurs at sampling instants. The sampling time sequence $\{t_k|_{k=1}^\infty\}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$. Let the time-varying digraph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)\}$ denote the communication graph of multi-agent network (1). It is assumed that the communication occurs only at sampling instants which implies that the communication graph $\mathcal{G}(t)$ do not contain any edge (i.e., $\mathcal{G}(t) = 0$) when $t \neq 0$. Due to technology limitations or communication constraints, it might be difficult to measure the relative or absolute velocity of agents [11], [14], [23], [24]. So the following impulsive algorithm without using any velocity information is proposed:

$$u_i(t) = -\sum_{k=1}^{+\infty}\left[p_1 \sum_{j\in N_i(t)} l_{ij}(t)\left(x_j(t) - x_i(t)\right) + p_2\left(x_i(t) - x_i(t - h_{k-1})\right)\right]\delta(t - t_k) \tag{2}$$

where $i \in \mathcal{V}$, $\delta(\cdot)$ is a Dirac function. Let $t_0 = 0$, $h_k = t_k - t_{k-1}$, when $t \in (t_{k-1}, t_k]$. Assume that the control gain $p_1 > 0$, $p_2 > 0$, $\underline{h} \leq h_k \leq \bar{h}$, and h_k belong to a finite set, where $\underline{h}, \bar{h}$ are positive constants and $k \in \mathbb{N}$.

Hence, multi-agent network (1) with impulsive algorithm (2) can be described by the following impulsive differential equations:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = 0, \quad t \in (t_k, t_{k+1}] \\ \Delta v_i(t_k) = -p_1 \sum\limits_{j\in V} l_{ij}(t_k) x_j(t_k) - p_2\left(x_i(t_k) - x_i(t_{k-1})\right), \end{cases} \tag{3}$$

where $\Delta v_i(t_k) = v_i(t_k^+) - v_i(t_k)$, $v_i(t_k^+) = \lim_{t\to t_k^+} v_i(t)$, $i \in \mathcal{V}$. It is assumed that $v_i(t)$ is left-hand continuous at $t = t_k$, $k \in \mathbb{N}$ and $v(t)$ is continuous at $t_0 = 0$.

Remark 1: In the proposed algorithm (2), the agent i, for any $i \in \mathcal{V}$, is required to obtain the sampled relative positions to its neighbors (i.e., $x_j(t_k) - x_i(t_k), j \in \mathcal{N}_i(t)$), and the sampled relative position to itself in previous sampling instant (i.e., $x_i(t_k) - x_i(t_{k-1})$). This is quite different from the existing consensus algorithms without velocity information [11], [14], [23], [24], which require continuous information of position.

Remark 2: Different from the existing consensus algorithms with sampled information [16]–[21], the sampling period in the multi-agent network (3) is not required to be a constant. Hence, the proposed algorithm can be applied to multi-agent network with time-varying or even uncertain sampling period.

Remark 3: For control input (2), the velocity state of each agent would be instantaneously changed. This kind of impulsive controller is useful and practical when the operating time of the impulsive controller is much smaller than the sampling interval. For example, in multi-robot systems, the velocity of each robot can be regulated to the desired state in a very short time, which can be described by the impulsive model.

III. Convergence Analysis

In this section, sufficient conditions on control gains p_1, p_2, control periods h_k, and the digraph $\mathcal{G}(t_k)$ will be developed so that consensus can be achieved under control input (2) with switching topology.

From (3), we have $v_i(t_{k+1}) = v_i(t_k^+)$

$$x_i(t_{k+1}) = x_i(t_k) + h_{k+1}v_i\left(t_k^+\right), \tag{4}$$

$$v_i\left(t_{k+1}^+\right) = (1-p_2h_{k+1})v_i\left(t_k^+\right) - p_1\sum_{j\in V} l_{ij}(t_{k+1}) \times x_j(t_k) - p_1h_{k+1}\sum_{j\in V} l_{ij}(t_{k+1})v_j\left(t_k^+\right). \tag{5}$$

Let $\tilde{x}_i(k) = x_i(t_k), \tilde{v}_i(k) = x_i(t_k) + \alpha v_i(t_k^+)$, where $\alpha = 2/p_2$. It is easy to show that the multi-agent network (1) achieves consensus, if $\lim_{k\to\infty}\tilde{x}_i(k) = \lim_{k\to\infty}\tilde{v}_i(k) = \beta$ where β is a constant. Then, it follows from (4) and (5) that:

$$\begin{cases} \tilde{x}_i(k+1) = \left(1-\frac{p_2h_{k+1}}{2}\right)\tilde{x}_i(k) + \frac{p_2h_{k+1}}{2}\tilde{v}_i(k), \\ \tilde{v}_i(k+1) = \frac{p_2h_{k+1}}{2}\tilde{x}_i(k) + \left(1-\frac{p_2h_{k+1}}{2}\right)\tilde{v}_i(k) \\ \qquad -(2/p_2 - h_{k+1})p_1\sum_{j\in\mathcal{V}} l_{ij}(t_{k+1})\tilde{x}_j(k) \\ \qquad -p_1h_{k+1}\sum_{j\in\mathcal{V}} l_{ij}(t_{k+1})\tilde{v}_j(k) \end{cases} \tag{6}$$

Let $\tilde{x}(k) = (\tilde{x}_1^T(k), \tilde{x}_2^T(k), \dots \tilde{x}_N^T(k))^T$, $\tilde{v}(k) = (\tilde{v}_1^T(k), \tilde{v}_2^T(k), \dots, \tilde{v}_N^T(k))^T$, then

$$\begin{pmatrix}\tilde{x}(k+1)\\ \tilde{v}(k+1)\end{pmatrix} = \mathbf{P}(k+1)\times\begin{pmatrix}\tilde{x}(k)\\ \tilde{v}(k)\end{pmatrix} \tag{7}$$

where

$$\mathbf{P}(k) = \begin{pmatrix}\mathbf{P}_1(k) & \mathbf{P}_2(k)\\ \mathbf{P}_3(k) & \mathbf{P}_4(k)\end{pmatrix} \tag{8}$$

$\mathbf{P}_1(k) = (1-p_2h_k/2)I$, $\mathbf{P}_2(k) = (p_2h_k/2)I$, $\mathbf{P}_3(k) = (p_2h_k/2)I - (2/p_2h_k)p_1L(t_k)$, and $\mathbf{P}_4(k) = (1-p_2h_k/2)I - p_1h_kL(t_k)$.

To proceed further, the following assumptions are made.

A1) There exists a positive integer l such that the union of $\mathcal{G}(t_k)$ across $k\in[k_0, k_0+l]$ contains a spanning tree, for any $k_0 > 0$.

A2) The control gains are chosen such that

$$p_1 < \min\left\{\frac{(p_2\underline{h})^2}{2\underline{h}(2-p_2\underline{h})\bar{l}}, \frac{2-p_2\bar{h}}{2\bar{h}\bar{l}}\right\} \tag{9}$$

where $\bar{l} = \sup_{i\in\mathcal{V},k\in\mathbb{N}}\{l_{ii}(t)\}$.

Lemma 6: If assumptions **A1**) and **A2**) hold, then $\mathbf{P}(k)$ given in (8), for any $k\in\mathbb{N}$, is a stochastic matrix with positive diagonal elements.

Proof: Note that $p_1, p_2, \underline{h}, \bar{h} > 0$, and when assumption **A1**) holds, $\bar{l} > 0$, which implies that $p_2\bar{h} < 2$ when assumptions **A1**) and **A2**) hold.

Then, it is easy to show that when $p_2\bar{h} < 2$, $\mathbf{P}_1(k)$ is a non-negative matrix with positive diagonal elements.

Obviously, $\mathbf{P}_2(k)$ is a non-negative matrix with positive diagonal elements.

It can be verified that when $p_2\bar{h} < 2$ and

$$p_1 < \frac{(p_2\underline{h})^2}{2\underline{h}(2-p_2\underline{h})\bar{l}}$$

$\mathbf{P}_3(k) \geq 0$ is a non-negative matrix with positive diagonal elements, for $k\in\mathbb{N}$.

It also can be verified that when

$$p_1 < \frac{2-p_2\bar{h}}{2\bar{h}\bar{l}}.$$

$\mathbf{P}_4(k)$ is a non-negative matrix with positive diagonal elements.

Hence, $\mathbf{P}(k)$ is nonnegative matrix with positive diagonal elements, when assumptions **A1**) and **A2**) hold. In addition, $\mathbf{P}(k)\mathbf{1}_{2N} = \mathbf{1}_{2N}$. Thus, $\mathbf{P}(k)$ is a stochastic matrix. □

Lemma 7: If assumptions **A1**) and **A2**) hold, then $\prod_{k=k_0}^{k_0+l}\mathbf{P}(k)$ is SIA.

Proof: According to the proof of Lemma 6, $\mathbf{P}_1(k)$, $\mathbf{P}_2(k)$, $\mathbf{P}_3(k)$, and $\mathbf{P}_4(k)$ are nonnegative matrix with positive diagonal elements when assumptions **A1**) and **A2**) hold. Hence, $\sum_{k=k_0}^{l+k_0}\mathbf{P}_1(k)$, $\sum_{k=k_0}^{l+k_0}\mathbf{P}_2(k)$, $\sum_{k=k_0}^{l+k_0}\mathbf{P}_3(k)$, and $\sum_{k=k_0}^{l+k_0}\mathbf{P}_4(k)$, are also non-negative matrices with positive diagonal elements.

Note that

$$\sum_{k=k_0}^{l+k_0}\mathbf{P}_1(k) = \left((l+1) - \frac{p_2}{2}\sum_{k=k_0}^{l+k_0}h_k\right)I,$$

$$\sum_{k=k_0}^{l+k_0}\mathbf{P}_2(k) = \left(\frac{p_2}{2}\sum_{k=k_0}^{l+k_0}h_k\right)I,$$

$$\sum_{k=k_0}^{l+k_0}\mathbf{P}_3(k) = \left(\frac{p_2}{2}\sum_{k=k_0}^{l+k_0}h_k\right)I - p_1\sum_{k=k_0}^{l+k_0}\left(\frac{2}{p_2} - h_k\right)L(t_k),$$

$$\sum_{k=k_0}^{l+k_0}\mathbf{P}_4(k) = \left((l+1) - \frac{p_2}{2}\sum_{k=k_0}^{l+k_0}h_k\right)I - p_1\sum_{k=k_0}^{l+k_0}h_kL(t_k).$$

Let $\tilde{\mathbf{P}} = \begin{pmatrix}(1-\alpha)I & \alpha I\\ \alpha I - \mu_1\tilde{L} & (1-\alpha)I - \mu_2\tilde{L}\end{pmatrix}$, where $\tilde{L} = \sum_{k=k_0}^{l+k_0}L(t_k)$, $0 < \alpha < 1$, $\mu_1 > 0$, $\mu_2 > 0$, $\alpha I - \mu_1\tilde{L}$ and $(1-\alpha)I - \mu_2\tilde{L}$ are nonnegative matrices with positive diagonal elements. Then, $\tilde{\mathbf{P}}$ is also a stochastic matrix and it is easy to check the edges in the digraph $\mathcal{G}(\tilde{\mathbf{P}})$ is the same as the digraph $\mathcal{G}(\sum_{k=k_0}^{l+k_0}\mathbf{P}(k))$.

Let λ be an eigenvalue of matrix $\tilde{\mathbf{P}}$, then $\det(\lambda I - \tilde{\mathbf{P}}) = 0$. Note that

$$\det(\lambda I - \tilde{\mathbf{P}}) = \prod_{i=1}^{N}\left(\lambda^2 - 2\lambda(1-\alpha) + \lambda\mu_2\gamma_i + (1-2\alpha) + (\alpha\mu_2 + \alpha\mu_1 - \mu_2)\gamma_i\right)$$

where $\gamma_i, i = 1, 2, \dots, N$ are the eigenvalues of $\tilde{L}$. Let $Q(\lambda) = \lambda^2 - 2\lambda(1-\alpha) + \lambda\mu_2\gamma_i + (1-2\alpha) - \mu_2\gamma_i + \alpha\mu_2\gamma_i + \alpha\mu_1\gamma_i$. Then, $Q(1) = \alpha\mu_2\gamma_i + \alpha\mu_1\gamma_i$. Hence, $\lambda = 1$ implies $\gamma_i = 0$ for some i. When $\gamma_i = 0$

$$Q(\lambda) = (\lambda - (1-2\alpha))(\lambda - 1). \tag{10}$$

The union of $\mathcal{G}(t_k)$ across $k\in[k_0, k_0+l]$ contains a spanning tree, so $\tilde{L}$ has one simple eigenvalue $\gamma_i = 0$. From $0 < \alpha < 1$, $1-2\alpha \neq 1$. Hence, from (10), $\tilde{\mathbf{P}}$ has one simple eigenvalue $\lambda = 1$. It follows from Lemma 4 that $\mathcal{G}(\tilde{\mathbf{P}})$ contains a spanning tree, which implies that $\mathcal{G}(\sum_{k=k_0}^{l+k_0}\mathbf{P}(k))$ contains a spanning tree.

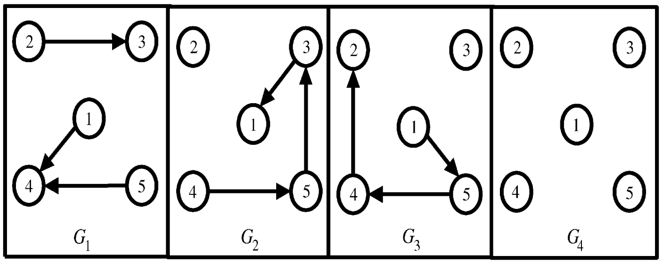

Fig. 1. Switching topology: $\mathcal{G}_i, i = 1, 2, 3, 4$.

It follows from Lemma 1 that:

$$\prod_{k=k_0}^{l+k_0} \mathbf{P}(k) \geq \varepsilon \sum_{k=k_0}^{l+k_0} \mathbf{P}(k) \tag{11}$$

for some $\varepsilon > 0$.

By Lemma 6, $\mathbf{P}(k)$ is a stochastic matrix with positive diagonal elements. Then, it is easy to show $\prod_{k=k_0}^{l+k_0} \mathbf{P}(k)$ is also a stochastic matrix. From (11), $\mathcal{G}(\prod_{k=k_0}^{l+k_0} \mathbf{P}(k))$ also contains a spanning tree. Then, it follows from Lemma 5 that the matrix $\prod_{k=k_0}^{l+k_0} \mathbf{P}(k)$ is SIA. $\square$

Now we are ready to present our main result as follows.

Theorem 1: If assumptions **A1**) and **A2**) hold, then the multi-agent network (1) with the impulsive algorithm (2) achieves consensus.

Proof: From Lemma 7, if assumptions **A1**) and **A2**) hold, $\prod_{k=k_0}^{l+k_0} \mathbf{P}(k)$ is SIA for any k_0. It follows from Lemma 2 that $\lim_{k\to\infty} \mathbf{P}(k)\cdots\mathbf{P}(1) = \mathbf{1}_{2N} y^T$ for some $y \in \mathbb{R}^{2N}$. From (7), $\lim_{k\to\infty} (\tilde{x}^T(k), \tilde{v}^T(k))^T = \mathbf{1}_{2N} y^T (\tilde{x}^T(0), \tilde{v}^T(0))^T$, which implies multi-agent network (1) achieves consensus. $\square$

Remark 4: Note that $p_2\bar{h} < 2$ when assumptions **A1**) and **A2**) hold. Then, the control gain $p_2 < 2/\bar{h}$, and it is easy to find a suitable p_1 according to (9).

Corollary 1: Consider the case of equidistant sampling interval, that is, $h_k = h$ with h being a constant. If assumption **A1**) holds and

$$p_1 < \min\left\{\frac{(p_2 h)^2}{2h(2 - p_2 h)\bar{l}}, \frac{2 - p_2 h}{2h\bar{l}}\right\}$$

where $\bar{l}$ is given by (10), then multi-agent network (1) with the impulsive algorithm (2) achieves consensus.

Corollary 2: Consider the case of fixed communication digraph, that is, $\mathcal{G}(t_k) = \mathcal{G}, k \in \mathbb{N}$. Let $L = (l_{ij})_{N\times N}$ be the Laplace matrix of $\mathcal{G}$. If $p_2 h_k < 2$ and

$$p_1 < \min\left\{\frac{(p_2 \underline{h})^2}{2\underline{h}(2 - p_2\underline{h}) \max\limits_{i\in\mathcal{V}}\{l_{ii}\}} \frac{2 - p_2\bar{h}}{2\bar{h} \max\limits_{i\in\mathcal{V}}\{l_{ii}\}}\right\}$$

then multi-agent network (1) with the impulsive algorithm (2) achieves consensus.

Corollary 3: Consider the case of both fixed communication digraph and the constant sampled period, that is, $\mathcal{G}(t_k) = \mathcal{G}$, $h_k = h$. $L = (l_{ij})_{N\times N}$ be the Laplace matrix of $\mathcal{G}$. If $\mathcal{G}$ contains a spanning tree and

$$p_1 < \min\left\{\frac{(p_2 h)^2}{2h(2 - p_2 h) \max\limits_{i\in\mathcal{V}}\{l_{ii}\}}, \frac{2 - p_2 h}{2h \max\limits_{i\in\mathcal{V}}\{l_{ii}\}}\right\}$$

then multi-agent network (1) with the impulsive algorithm (2) achieves consensus.

IV. Illustrative Examples

In this section, a numerical example is given to illustrate the effectiveness of the proposed algorithm. Consider a multi-agent network (1) with the switching topology as shown in Fig. 1. The corresponding Lapacian matrices of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ are

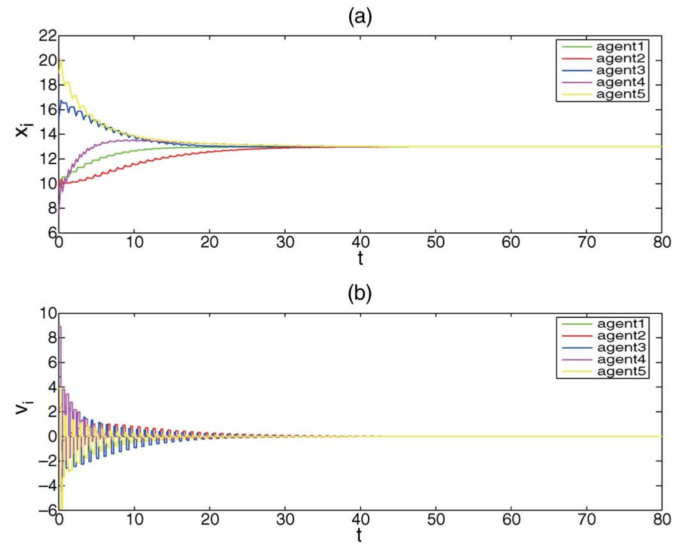

Fig. 2. Trajectory of the multi-agent network (1) under switched topology, for $p_1 = 0.4$ and $p_2 = 5$. Evolution of (a) x_i, and (b) v_i.

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $L_4 = 0$, respectively. Assume that $h_1 = 0.2, h_2 = 0.3, h_3 = 0.2, h_4 = 0.3, \ldots, \mathcal{G}(t_1) = \mathcal{G}_1, \mathcal{G}(t_2) = \mathcal{G}_2, \mathcal{G}(t_3) = \mathcal{G}_3, \mathcal{G}(t_4) = \mathcal{G}_4, \mathcal{G}(t_5) = \mathcal{G}_1, \mathcal{G}(t_6) = \mathcal{G}_2, \cdots$. Then, $\underline{h} = 0.2$, $\bar{h} = 0.3$, and $\sup_{i\in\mathcal{V},k\in\mathbb{N}}\{l_{ii}(t_k)\} = 2$. According to Remark 4, choose $p_2 = 5 < 2/\bar{h} \approx 6.667$. It follows from (9) that $p_1 < \min\{1.25, 0.4167\}$. Choose $p_1 = 0.4$. Fig. 2 shows that consensus can be achieved with chosen control gains $p_1 = 0.4$ and $p_2 = 0.5$. However, it is noted that when one control gains p_1 and p_2 is chosen to be too large so that the sufficient condition is not satisfied, consensus can not be achieved as shown in Fig. 3 and Fig. 4.

V. Conclusion

In this technical note, the consensus problem has been studied for continuous-time second-order multi-agent networks with switching topology under aperiodic sampled communication based on position-only information. Some sufficient conditions have been obtained to ensure consensus of multi-agent networks. A numerical example has been given to demonstrate the effectiveness of the proposed algorithm. It would be interesting to further investigate the multi-agent networks with quantized communication via hybrid impulsive algorithms to realize network consensus.

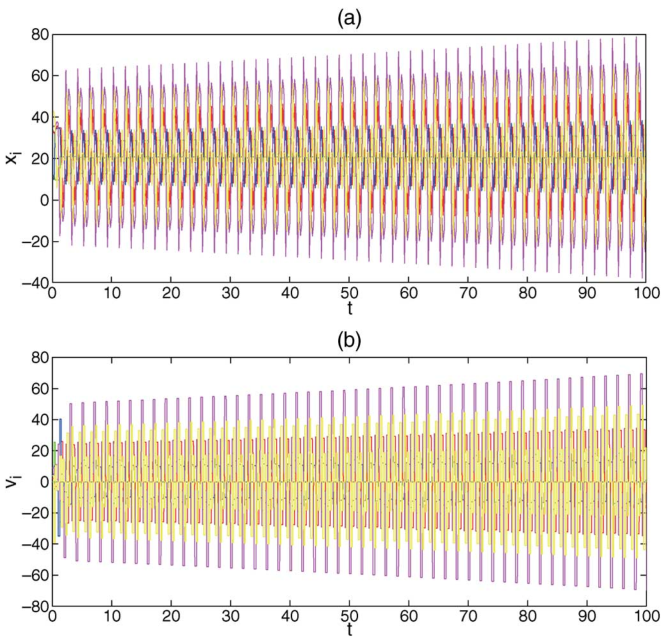

Fig. 3. Trajectory of the multi-agent network (1) under switched topology, for $p_1 = 3.77$ and $p_2 = 5$. Evolution of (a) x_i, and (b) v_i.

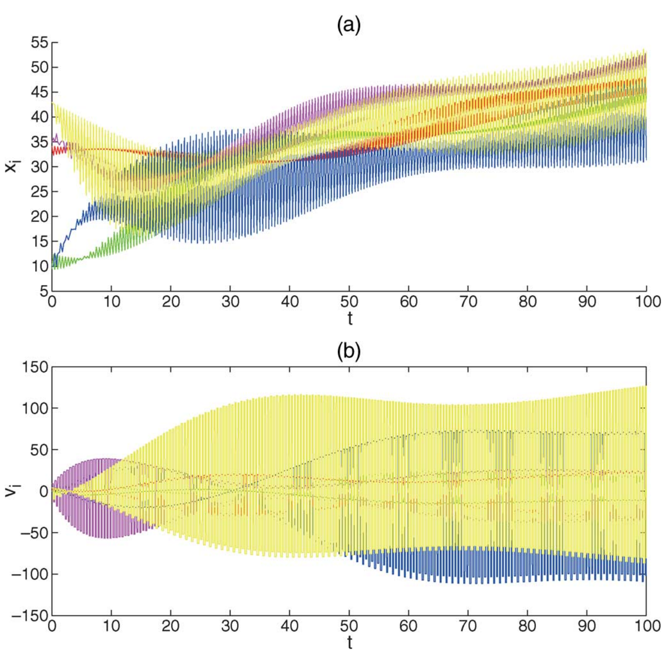

Fig. 4. Trajectory of the multi-agent network (1) under switched topology, when $p_1 = 0.4$ and $p_2 = 8.2$. Evolution of (a) x_i, and (b) v_i.

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