different controllable subspaces, we ask if arbitrary rational inputs do not
generate an even larger controllable subspace. (Of course, in this case,
one may not hope to have smooth trajectories.)

Definition 4.1: A regular system $(E, A, B)$ will be said to be almost
controllable if, given any initial condition $x(0)$, there exists a finite
time $T > 0$, a $U(s) \in R^n(s)$, and a unique $X(s) \in R^n(s)$ so that $(sE - A) - B(U(s) = Ex(0), s) x(T) = 0$. A regularizable system will be
said to be "almost controllable" if $(E, A, B)$ is almost controllable for
some $E$ and $K$.

Theorem 4.3: A regularizable system $(E, A, B)$ is "almost control-
lar" iff $R^E \subset R^n$.

Proof: Given a regularizable system $(E, A, B)$, we first choose a $K$
so that: i) $KEF_{x}x^{*} = 0$ and ii) $IMB \subset E_{x}V^{*}$. Note that regu-
larizability of $(E, A, B)$, Theorem 3.2, and Lemma 2.2 guarantee the
existence of such a $K$. Then, $(E, A, B)$ is regular with $V^{*}$ as its
initial manifold and therefore, exist two nonsingular matrices $W$ and
$Y$ so that the Weierstrass form $(W^{-1}EXV, V^{-1}AV, W^{-1}W^{-1}B)$ [1] of
$(E, A, B)$ becomes [5]

$$x_{1} = Jx_{1} + B_{1}u$$

$$N_{x_{1}} = x_{2}$$

where $N$ is a Jordan-form matrix corresponding to the zero eigen-
value. Clearly, $(E, A, B)$ is "almost controllable" iff $(E, A, B)$ is
"almost controllable" and, as it is regular, $(E, A, B)$ is "al-
most controllable" iff it is almost controllable, or equivalently iff
$(W^{-1}EXV, V^{-1}AV, W^{-1}B)$ is almost controllable. As $x_{1}(t) = 0$
for $t > 0$, almost controllability of $(W^{-1}EXV, V^{-1}AV, W^{-1}B)$ is
equivalent to the controllability of the state-space system $(J, B)$. How-
ever, $(J, B)$ is controllable iff $R^E = R^n$.

Lemma 4.2: Let $(E, A, B)$ be regularizable and let $y \in R^n$ be given.
If there exist $T > 0, X(s) \in R^n(s)$ and $U(s) \in R^n(s)$ satisfying:

a) $(sE - A) - B(U(s) = 0$,

b) $x(T) = y$ (where, as before, $x(t)$ denotes the inverse Laplace

transform of $X(s)$), then there exist $X_{1}(s) \in R_{x}(s)$ and $U(s) \in R_{x}(s)$

enjoying the same properties a) and b).

This result demonstrates that if a point $y$ can be reached from the origin
in finite time along an impulsive trajectory generated by an impulsive
input, then the same point can be reached from the origin in finite time
along a smooth trajectory generated by a smooth input. This explains
why enlarging the class of inputs from $R^n(s)$ to $R^n(s)$ (or, equivalently,
introducing derivative feedback) does not enlarge the reachable space
of the system; and why there is no need to introduce the concept of "almost
reachtability."

Proof (Lemma 4.2): First, choose an $F$ so that $(E, A, B)$ is regu-
lar. Let $x, T, U(s), x(s)$ be as in the statement of the lemma. Expand $x(s)$ and $U(s)$ as in (4.2) and (4.3). It is easy to show that

$$x_{k} = x_{k}^{*} \forall k \leq 0 \text{ and } x_{k} \in V^{*} \forall k > 0$$

The initial condition is zero, we also have $E_{x}x_{k} + B_{x}u_{k}$. Then,

$$x_{k} \in R_{x} \cap A^{-1}(E_{x}V^{*} + IMB) \cap V^{*} = R^{*} \cap V^{*} = R^{*}$$

It follows that $y = x(T)$ is in $R^{*}$. As $(E, A, B)$ is regular and as

$$y \in R^{*}$$

there exists a smooth input $u(t)$ with a strictly proper Laplace transform $U(s)$ so
that the trajectory $x(t)$ is smooth with strictly proper Laplace transform $X(s)$ which satisfies $x(t) = x[5]$. Then, taking $x_{1}(s) = X(s)$ and $U(s) = x(s) = Fx(s)$ completes the proof.

V. CONCLUSIONS

Arguing against the assumption of regularity which overwhelms the literature on continuous-time singular systems, we have introduced

the notion of regularizability and have shown that, unlike regularity, it is
invariant under linear "state" feedback. We have established that a modi-
fication of the definition of reachability so as to make both on regularizability rather than regularity is not only possible but also desir-
able as it is invariant under linear "state" feedback. It has also been
shown that a similar remedy of the noninvariance of controllability under
linear feedback turns out to be somewhat involved in the sense that the
correct way to define controllability depends on the type of feedback
law to be used. Thus, we have defined "controllability by proportional
feedback," "controllability by derivative feedback," and "controlla-
bility by proportional-plus-derivative feedback" and have shown that the last
two are the one and the same property. Apart from their feedback and
geometric characterizations, dynamical interpretations of these concepts
have also been introduced. It has been shown that, under the quite nat-
ural assumption that $ImE = ImA + ImB = R^{*}$, reachability condi-
tions whereby given any initial condition one can find at least one
admissible input which generates a trajectory. The dependence of the de-

inition of controllability on the type of feedback has been rewritten as
a symbol of its dependence on the type of inputs to be used to drive the
given initial condition to the origin. It has been established that the use
of derivative feedback in the closed-loop system is equivalent to using an
open-loop control which has a Dirac delta term. Finally, the definition
of reachability has been shown to be insensitive not only to the changes
in the type of feedback inputs, but also to possible changes in the class
of open-loop inputs.

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Optimal Reduced-Order Solution of the Weakly Coupled Discrete Riccati Equation

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Abstract—The optimal solution of the weakly coupled algebraic dis-
crete Riccati equation is obtained in terms of a reduced-order continuous

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type algebraic Riccati equation via the use of a bilinear transformation. The proposed method has a rate of convergence of $O(\epsilon^2)$ where $\epsilon$ represents a small coupling parameter. The method is applicable under mild
assumptions.

I. INTRODUCTION

Linear weakly coupled continuous systems have been studied in [11]-[12]. However, linear weakly coupled discrete systems have not been studied due to the fact that the partitioned form of the main equation of the optimal linear control theory, Riccati equation, has a very complicated form in the discrete-time domain. This note overcomes that problem by the use of a bilinear transformation, which is applicable under mild assumptions, such that the solution of the discrete algebraic Riccati equation of weakly coupled systems is obtained using results from the corresponding continuous-time equation.

The algebraic Riccati equation of weakly coupled linear discrete systems is given by

$$P = A^TPA + Q - A^TPBR(B^TPB + R)^{-1}B^TPA$$

where

$$A = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_3 & Q_4 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

and $\epsilon$ is a small coupling parameter. Due to the block dominant structure of the problem matrices, the required solution $P$ has the form

$$P = \begin{pmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & P_3 \end{pmatrix}.$$  (2)

The main goal in weakly coupled system theory is to obtain the required solution in terms of reduced-order problems, namely, subsystems. In the case of the weakly coupled algebraic discrete Riccati equation, the inversion of the partitioned matrix $(B^TPB + R)$ will produce a lot of nonzero terms and make the corresponding approach computationally very involved, even though one is faced with reduced-order numerical problems.

To solve this problem, we have used a bilinear transformation introduced in [13] to transform the discrete Riccati equation (1) into a continuous-time algebraic Riccati equation of the following form:

$$A^TP + P_1A_1 + Q_1 - P_3S_3P_3 = 0, \quad S_1 = B_1R_1^{-1}B_1^T.$$  (3)

The solution of (1) is equal to the solution of (3). Appendix I shows that (3) preserves the structure of weakly coupled systems and can be efficiently solved in terms of the reduced-order fixed-point method developed in [7]. The required solution is then obtained with the rate of convergence of $O(\epsilon^2)$.

II. COMPUTATIONAL ALGORITHM

Since the proposed algorithm for a discrete algebraic Riccati equation combines features of the bilinear transformation [13] and the fixed-point algorithm developed in [7] for the weakly coupled continuous algebraic Riccati equation, we will briefly summarize the main results obtained in [13] and [7].

The bilinear transformation states that (1) and (3) have the same solution if the following relations hold [13]:

$$A_e = I - 2D^{-T}$$

$$S_e = 2(I + A)^{-1}S_dA^{-1}, \quad S_d = BR^{-1}B^T$$

$$Q_e = 2D^{-1}Q(I + A)^{-1}$$

$$D = (I + A^T)^{-1} + (Q + Q_e)^{-1}S_d$$

and assuming that $(I + A)^{-1}$ exists. Matrix $D$ has been shown to be invertible [15]. The physical interpretation of the transformation between the continuous and discrete type algebraic Riccati equation is discussed in [13].

The proposed algorithm will be valid under the following assumption.

**Assumption 1:** The system matrix $A$ has no eigenvalues located at $-1$.

It is important to point out that the eigenvalues located in the neighborhood of $-1$ will produce ill conditioning with respect to matrix inversion and make the algorithm numerically unstable.

It can be verified that the weakly coupled structure of matrices defined in (1) will produce the weakly coupled structure of transformed matrices given in (4) (Appendix I). The compatible partitions of these matrices are

$$A_e = \begin{pmatrix} A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}, \quad S_e = \begin{pmatrix} S_{11} & \epsilon S_{12} \\ \epsilon S_{21} & S_{22} \end{pmatrix},$$

$$Q_e = \begin{pmatrix} Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & Q_{22} \end{pmatrix}, \quad P_e = P_1 + \epsilon P_2.$$

These partitions have to be performed by a computer only in the process of calculations and there is no need for the corresponding analytical expressions.

Solution of (3) can be found in terms of the reduced-order problems by imposing standard stabilizability-detectability assumptions on the subsystems. The efficient fixed-point reduced-order algorithm for solving (3) is obtained in [7].

The $O(\epsilon^2)$ approximation of (3) is obtained from the following decoupled set of equations:

$$P_1A_{11} + \epsilon A_{12}P_1 + Q_{11} - P_3S_3P_3 = 0$$  (5a)

$$P_1A_{21} + \epsilon A_{22}P_1 + Q_{21} - P_3S_3P_3 = 0$$  (5b)

$$P_2(A_{22} - S_3P_3) = (A_{22} - S_1P_1)P_2 + P_1A_{21} + \epsilon A_{22}P_3$$

$$P_3 = (I + A_{22})^{-1} + (P_3 + Q_{22}P_3)^{-1}S_3P_3 = 0.$$  (5c)

The unique positive semidefinite stabilizing solutions of (5a) and (5b) exist under the following assumption.

**Assumption 2:** Triples $(A_{1i}, \sqrt{S_{1i}}, \sqrt{Q_{1i}}), i = 1, 2$ are stabilizable-detectable.

Defining the approximation errors as

$$P_i - P_i^j = \epsilon^j E_i^j, \quad i = 1, 2, 3$$  (6)

the fixed-point type algorithm, with the rate of convergence of $O(\epsilon^2)$, is obtained in [7] in the decoupled form as

$$E_i^{(j+1)} \Delta_1 + \epsilon \Delta_1^j E_i^{(j+1)} = M_{1i}^{(j)}$$

$$E_i^{(j+1)} \Delta_2 + \epsilon \Delta_1^j E_i^{(j+1)} = M_{2i}^{(j)}$$

$$E_i^{(j+1)} \Delta_2 + \epsilon \Delta_1^j E_i^{(j+1)} = M_{3i}^{(j)}$$

$$E_i^{(j+1)} \Delta_2 + \epsilon \Delta_1^j E_i^{(j+1)} = M_{4i}^{(j)}.$$  (7c)

with $j = 0, 1, 2, \cdots$, and $E_i^{(0)} = 0, \epsilon \Delta_1 = 0, \epsilon \Delta_2 = 0$ where newly defined matrices are given in Appendix II. Note that $\Delta_1$ and $\Delta_2$ are stable matrices [11].

The rate of convergence of (7) is $O(\epsilon^2)$ [7], that is

$$\|P_i - P_i^j\| = O(\epsilon^2), \quad i = 1, 2, 3; \quad j = 0, 1, 2, \cdots.$$  (8)

where

$$P_i = P_i + \epsilon E_i^{(j)}, \quad i = 1, 2, 3; \quad j = 0, 1, 2, \cdots.$$  (9)

The proposed algorithm for the reduced-order solution of the discrete algebraic Riccati equation under conditions stated in Assumptions 1 and 2 has the following form.

1) Transform (1) into (3) using (4).
2) Solve (3) using the reduced-order algorithm (5)-(7).
TABLE I
REDUCED-ORDER SOLUTION OF THE DISCRETE WEAKLY COUPLED
ALGEBRAIC RICCATI EQUATION

<table>
<thead>
<tr>
<th>j</th>
<th>( P_1(j) )</th>
<th>( P_2(j) )</th>
<th>( P_3(j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>39.937 2.6157</td>
<td>3.5666 2.5105 30.533</td>
<td>1.4898 0.3050 2.1505</td>
</tr>
<tr>
<td>1</td>
<td>51.477 3.7414</td>
<td>4.4048 2.6952 34.448</td>
<td>1.5646 0.3468 2.5341</td>
</tr>
<tr>
<td>2</td>
<td>56.881 4.3019</td>
<td>4.7920 1.5161 53.815</td>
<td>1.5985 0.3665 2.6762</td>
</tr>
<tr>
<td>3</td>
<td>60.179 4.6541</td>
<td>5.0293 2.6257 36.500</td>
<td>1.6186 0.3763 2.7440</td>
</tr>
<tr>
<td>4</td>
<td>60.733 4.7144</td>
<td>5.0644 3.2826 36.600</td>
<td>1.6218 0.3778 2.7520</td>
</tr>
<tr>
<td>5</td>
<td>60.824 4.7243</td>
<td>5.0708 3.2864 36.616</td>
<td>1.6223 0.3781 2.7542</td>
</tr>
<tr>
<td>6</td>
<td>60.839 4.7258</td>
<td>5.0714 3.2865 36.617</td>
<td>1.6224 0.3781 2.7544</td>
</tr>
</tbody>
</table>

III. NUMERICAL EXAMPLES
A real world physical example (a chemical plant model [14]) demonstrates the efficiency of the proposed method

\[
\begin{align*}
A &= 10^{-2} \begin{bmatrix}
95.407 & 1.9643 & 0.3597 & 0.0673 & 0.0190 \\
40.849 & 41.317 & 16.084 & 4.4679 & 1.1971 \\
4.1118 & 12.858 & 27.209 & 21.442 & 40.976 \\
0.1305 & 0.5808 & 1.8750 & 3.6162 & 94.280
\end{bmatrix} \\
B^T &= 10^{-2} \begin{bmatrix}
0.0434 & 2.6606 & 3.7530 & 3.6076 & 0.4617 \\
-0.0122 & -1.0453 & -5.5100 & -6.6000 & -0.9148
\end{bmatrix}
\end{align*}
\]

\[
Q = I_1, \quad R = I_1
\]

These matrices are obtained from [14] by performing a discretization with a sampling rate \( \Delta T = 0.5 \). The small weakly coupling parameter \( \epsilon \) is built into the problem and can be roughly estimated from the strongest coupled matrix (matrix \( B \)). The strongest coupling is in the third row, where

\[
\epsilon = \frac{s_{31}}{s_{11}} = \frac{3.7530}{3.6076} \approx 0.68.
\]

Simulation results are obtained using the L-A-S package for computer-aided control system [16] and presented in Table I.

For this specific real world example the proposed algorithm perfectly matches the presented theory since convergence, with the accuracy of \( 10^{-4} \), is achieved after 9 iterations (i.e., \( 0.068^9 = 10^{-4} \)). Numerical examples performed in [7] for different values of \( \epsilon \) support the proposed algorithm.

IV. CONCLUSION
A reduced-order optimal solution of the algebraic discrete weakly coupled Riccati equation is obtained. This result reduces off-line computa-
tional requirements and plays an important role in the design procedure of optimal and near-optimal controllers and filters for weakly coupled discrete systems.

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