An Eigenvalue Based Carrier Frequency Offset Estimator for OFDM Systems

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***Abstract*—Orthogonal frequency division multiplexing (OFDM) is sensitive to frequency synchronization errors. This letter proposes a novel data-aided carrier frequency offset (CFO) estimator. We show that the eigenvalues of the inter-carrier interference (ICI) coefficient matrix are the elements of a geometric series distributed on the unit circle of the complex plane. Then, we prove that estimating the CFO is equivalent to finding the eigenvalues of a two-dimensional ICI coefficient matrix. As a result, by transmitting the corresponding eigenvectors, an estimate of the CFO value can be found. In addition to its simplicity, the proposed estimator is proven to be a maximum likelihood estimator. Simulation results are presented to demonstrate the high accuracy of the proposed estimator in presence of channel noise and fading.**

***Index Terms*—Orthogonal frequency division multiplexing, frequency synchronization errors, carrier frequency offset estimation.**

I. Introduction

CURRENTLY, orthogonal frequency division multiplexing (OFDM) has been adopted in many standards such as asymmetric digital subscriber line (ADSL), digital video broadcasting (DVB), and wireless local area networks (WLANs) [1], [2]. Also, OFDM based multiple access technology (OFDMA) has been adopted in the Long Term Evolution (LTE) fourth generation mobile broadband communication standard [1]. This is mainly due to OFDM high spectral efficiency and its ability to deal with severe propagation delay dispersion without complex channel equalizers. However, OFDM systems are highly sensitive to frequency synchronization errors, which are referred to as carrier frequency offset (CFO) [3]. These synchronization errors can be due to a mismatch between the transmitter and receiver local oscillator frequencies or as a result of the Doppler effect. The CFO violates the OFDM sub-carriers (SCs) orthogonality and hence the received signal suffers from attenuation, phase rotation, and inter-carrier interference (ICI) from other SCs in the OFDM signal [3], leading to detection errors.

In literature, CFO mitigation techniques can be broadly categorized into two groups. The first group includes the CFO estimation and correction techniques [4] - [12] and the second group includes the CFO sensitivity reduction techniques [13] - [15]. In CFO estimation and correction techniques, the CFO is estimated and corrected at the receiver side. In general, CFO estimation can be divided into two main categories, namely data-aided [4] - [9] and blind [10] - [12] methods. One drawback of these algorithms is their computational complexity which is expressed in terms of complex multiplications and additions. Finally, CFO reduction techniques aim to reduce the sensitivity of OFDM systems against the CFO. These include polynomial cancellation coding [13], [14] and DFT-based cancellation [15].

In this letter, we focus on CFO estimation and correction techniques. The objective is to develop a CFO estimator with low computational complexity and high estimation accuracy. Through mathematical analysis, we show that the eigenvalues of the ICI coefficient matrix S, with dimension $N \times N$, are the elements of a geometric series distributed on the unit circle of the complex plane. As a result, for two OFDM systems with ICI coefficient matrices, S_1 with dimension $N_1 \times N_1$ and S_2 with dimension $N_2 \times N_2$ and $N_1 < N_2$, with the same CFO value, the eigenvalues of S_1 are included as a subset of the eigenvalues of S_2. Then, we prove that for an OFDM system with $N >> 2$ SCs, the CFO information can be recovered from the eigenvalues of a two-dimensional ICI coefficient matrix. Hence, by transmitting the correspondinge eigenvectors, the CFO estimation and correction can be performed at the receiver. The main attractive features of the proposed estimator are its simplicity and high estimation accuracy under different channel conditions. The proposed CFO estimator is proven to be a maximum likelihood estimator (MLE).

The rest of this letter is organized as follows. The system model is described in Section II. The CFO estimator is presented in Section III. The CFO attractive features are discussed in Section IV. Simulation results are given in Section V. Finally, conclusions are drawn in Section VI.

II. System Model

A baseband discrete-time OFDM system with N SCs is considered. The OFDM symbol is generated by taking the N-point inverse discrete Fourier transform (IDFT) on the information symbol $X(k)$, $0 \leq k < N$. Hence, the OFDM symbol at the transmitter side is given by

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}, \qquad n = 0, 1, \ldots, N-1 \quad (1)$$

where $j = \sqrt{-1}$.

After passing through the channel, the received signal at baseband is expressed as [6]

$$y(n) = e^{j\frac{2\pi\varepsilon n}{N}} \cdot [h(n) \otimes x(n)] + w(n), \qquad n = 0, 1, \ldots, N-1 \quad (2)$$

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where $h(n)$ is the baseband channel impulse response, $\otimes$ denotes convolution, ε is the normalized CFO (the ratio of the carrier frequency offset to the SCs spacing), and $w(n)$ is the complex additive white Gaussian noise (AWGN).

For demodulation, DFT is applied to the received signal, which results in

$$\begin{aligned} Y(k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y(n) e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1 \\ &= \frac{1}{N} \sum_{l=0}^{N-1} \{[\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(l-k+\varepsilon)}] H(l) X(l)\} + W(k), \\ &\qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{3}$$

where $H(l)$ is the transfer function of the channel at the frequency of the l^{th} SC and $W(k)$ is a zero mean complex Gaussian noise. Let $S(l-k)$ denote the ICI-coefficient of SC l on SC k [3], which is given by

$$S(l-k) = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(l-k+\varepsilon)}, \quad k, l = 0, 1, \ldots, N-1. \tag{4}$$

After some algebraic manipulation, $S(l-k)$ is reduced to

$$S(l-k) = \frac{\sin(\pi(l-k+\varepsilon))}{N \sin(\pi \frac{l-k+\varepsilon}{N})} e^{j\pi(l-k+\varepsilon)(1-\frac{1}{N})}. \tag{5}$$

Hence, $Y(k)$ can be written as

$$Y(k) = \sum_{l=0}^{N-1} S(l-k) H(l) X(l) + W(k), \quad k = 0, 1, \ldots, N-1. \tag{6}$$

Equation (6) can be re-written as

$$\begin{aligned} Y(k) &= S(0) H(k) X(k) + \\ &\sum_{l=0, l\neq k}^{N-1} S(l-k) H(l) X(l) + W(k), \ k = 0, \ldots, N-1. \end{aligned} \tag{7}$$

From (7), we can see that the original symbol ($X(k)$) suffers at reception from attenuation ($\frac{\sin(\pi\varepsilon)}{N\sin(\pi\frac{\varepsilon}{N})}$), phase rotation ($e^{j\pi\varepsilon(1-\frac{1}{N})}$), and ICI from other symbols ($S(l-k)$). This is due to the CFO ($\varepsilon \neq 0$). In order to compensate for these effects, the CFO value, ε, needs to be estimated and removed at the receiver.

In a matrix form, (6) can be written as

$$Y = SHX + W \tag{8}$$

where $Y = [Y(0)Y(1) \ldots Y(N-1)]^T$, S is an $N \times N$ square matrix with elements $S(l-k)$, H is a diagonal matrix with elements $H(k)$, $X = [X(0)X(1) \ldots X(N-1)]^T$, $W = [W(0)W(1) \ldots W(N-1)]^T$, and T denotes the transpose operation.

III. CFO Estimation Algorithm

Let $d = e^{-j\frac{2\pi}{N}}$. Define D as the DFT matrix, which is given by

$$D = \frac{1}{\sqrt{N}} \cdot \begin{bmatrix} d^0 & d^0 & d^0 & \ldots & d^0 \\ d^0 & d^1 & d^2 & \ldots & d^{N-1} \\ d^0 & d^2 & d^4 & \ldots & d^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d^0 & d^{N-1} & d^{2(N-1)} & \ldots & d^{(N-1)^2} \end{bmatrix}. \tag{9}$$

Using (4), the ICI-coefficient matrix, S, can be written as

$$S = DAD^* \tag{10}$$

where $*$ denotes the conjugate transpose and A is a diagonal matrix with elements $a(n,n) = e^{j\frac{2\pi\varepsilon n}{N}}$, $n = 0, 1, \ldots, N-1$. Since $DD^* = I$ (where I is the identity matrix), D and D^* are unitary [16]. Hence, from (10), S and A are similar matrices [16]. As a result, S and A have the same eigenvalues. Also, the corresponding eigenvectors are given by the columns of D^*.

Using the diagonal elements of matrix A, the eigenvalues of matrix S are given by $E = \{1, e^{j\frac{2\pi\varepsilon}{N}}, \ldots, e^{j\frac{2\pi(N-1)\varepsilon}{N}}\}$. Hence, the eigenvalues of the ICI coefficient matrix S are the elements of a geometric series distributed on the unit circle of the complex plane. Since the normalized CFO value, ε, appears in every element of $E(n)$ for $n = 1, 2, \ldots, N-1$, it is sufficient to use only two SCs in CFO estimation. This results in an improved spectral efficiency in the CFO estimation process, as will be shown later.

For a large number N of SCs, the adjacent SCs experience approximately the same channel response [5]. Consider an OFDM system with only two SCs with the same frequency spacing as in the original N SCs system (i.e. $\tilde{N} = 2$). The channel matrix elements, $\tilde{H}(0)$ and $\tilde{H}(1)$, can be approximately equal, i.e. $\tilde{H}(0) \approx \tilde{H}(1) = H_0$. Hence, the channel matrix for the estimation SCs ($\tilde{N} = 2$) is given by $\tilde{H} = H_0 I$. As a result, (8) is reduced to

$$Y = H_0 S X + W \tag{11}$$

where the ICI coefficient matrix S now has a dimension of $\tilde{N} \times \tilde{N}$. Let X_n denote the $\tilde{N}$ dimensional eigenvector corresponding to the n^{th} eigenvalue $E(n)$, $n = 0, 1$. By transmitting X_n, (11) can be written as

$$\begin{aligned} Y &= H_0 S X_n + W \\ &= H_0 E(n) X_n + W \\ &= H_0 e^{j\pi n\varepsilon} X_n + W. \end{aligned} \tag{12}$$

To remove the effect of the AWGN, we transmit $X_0 = [11]^T$ for M times. The average received vector $\bar{Y}$ is given by $[z_1 z_2]^T = H_0 \cdot [11]^T$, for a zero mean AWGN. Hence, an estimate of H_0 is given by

$$\hat{H}_0 = \frac{1}{2}(z_1 + z_2). \tag{13}$$

Similarly, by transmitting $X_1 = [1 \ -1]^T$ for M times, the average received vector $\bar{Y}$ is given by $[z_3 z_4]^T = H_0 e^{j\pi\varepsilon} \cdot [1 \ -1]^T$, for a zero mean AWGN. Hence, an estimate of ε is given by

$$\hat{\varepsilon} = \frac{(\angle(z_3 - z_4) - \hat{\theta})}{\pi} \tag{14}$$

where $\hat{\theta} = \angle\hat{H}_0 = \angle(z_1 + z_2)$. As a result,(14) is reduced to

$$\hat{\varepsilon} = \frac{(\angle(\frac{z_3-z_4}{z_1+z_2})}{\pi}. \tag{15}$$

The size of the training symbol used for the CFO estimation is given by $4M$ (i.e. transmitting each of X_0 and X_1, which has a size of 2, for M times). The larger the number of transmissions, M, the lower the estimation error and the larger the training symbol size. The trade-off between the estimation accuracy and the spectral efficiency is further investigated in the simulation result section.

The CFO estimation algorithm is described in Algorithm 1[1].

Algorithm 1 An Eigenvalue Based CFO Estimation

$Y_0 = 0$, $Y_1 = 0$;
for $m = 1 : M$ **do**
  Transmit the eigenvector X_0 and receive Y_{0m};
  At receiver, $Y_0 = Y_0 + Y_{0m}$;
  Transmit the eigenvector X_1 and receive Y_{1m};
  At receiver, $Y_1 = Y_1 + Y_{1m}$;
end for
$\bar{Y}_0 = \frac{1}{M}Y_0 = [z_1z_2]^T$;
$\bar{Y}_1 = \frac{1}{M}Y_1 = [z_3z_4]^T$;
$\hat{\varepsilon} = \frac{(\angle(\frac{z_3-z_4}{z_1+z_2})}{\pi}$;

IV. CFO Estimator Advantages

In this section, the main attractive features of the proposed CFO estimator are discussed in terms of high estimation accuracy and low computational complexity.

A. High Estimation Accuracy

For clarity of presentation, consider a slow fading channel scenario, where the impulse response of the channel does not change (much) during the M times transmission of X_0 and X_1 [4]. The high estimation accuracy is proven by showing that the proposed estimator is an MLE.

From Algorithm 1, when X_0 is transmitted, the average received vector is given by $[z_1z_2]^T = H_0{\cdot}[11]^T + \frac{1}{M}\sum_{m=1}^{M} W_m^T$. It can be shown that the MLE of H_0 is given by minimizing the quadratic term

$$Q_1 = z_1^*z_1 + z_2^*z_2 + 2H_0^*H_0 - H_0^*z_1 - H_0z_1^* - H_0z_2^* - H_0^*z_2. \tag{16}$$

From (16), the resulting MLE of H_0 is

$$\check{H}_0 = \frac{1}{2}(z_1 + z_2) \tag{17}$$

which is equivalent to our estimate in (13). The Cramer-Rao bound of the estimation is easily proven to be

$$\text{CRB}_H = \frac{1}{4M\gamma} \tag{18}$$

where γ is the signal-to-noise ratio (SNR) of the received symbol $\bar{Y}$.

Similarly, when X_1 is transmitted, the average received vector is given by $[z_3z_4]^T = H_0e^{j\pi\varepsilon}{\cdot}[1\ -1]^T + \frac{1}{M}\sum_{m=1}^{M} W_m^T$. It can be shown that the MLE of H_0 is given by minimizing the quadratic term

$$Q_2 = z_3^*z_3 + z_4^*z_4 + 2H_0^*H_0 - H_0^*e^{-j\pi\varepsilon}z_3 - H_0e^{j\pi\varepsilon}z_3^* \\ - H_0e^{-j\pi\varepsilon}z_4^* - H_0^*e^{j\pi\varepsilon}z_4. \tag{19}$$

From (19), the resulting MLE of ε is

$$\check{\varepsilon} = \frac{(\angle(\frac{z_3-z_4}{z_1+z_2})}{\pi} \tag{20}$$

which is equivalent to our estimate in (15). The Cramer-Rao bound of the estimation is easily proven to be

$$\text{CRB}_\varepsilon = \frac{1}{2M\pi^2\gamma}. \tag{21}$$

B. Low Computational Complexity

The proposed CFO estimator exhibits a low computational complexity as compared with other existing estimation algorithms. The CFO value is directly given by finding the angle of a ratio of the average received signals. Hence, the estimator requires no complex multiplication operations unlike the existing algorithms such as [4], which requires N complex multiplications, and [5], which requires $N - p - 1$ complex multiplications for a small integer p, e.g. $p = 2$. In addition, the new CFO estimator does not require solving any optimization problem as in [6] and references therein.

In a static scenario, when the CFO value does not change much over a communication session, the training symbol (transmitting X_0 and X_1 for M times) can be employed only once at the beginning of the session for CFO estimation. The process can be repeated according to the reception quality. In a dynamic scenario, with a time varying CFO value, the training symbol can be included within the preamble of each data frame[2].

V. Simulation Results and Discussion

In this section, we show simulation results for the proposed CFO estimator in AWGN and Rayleigh fading channels. The average channel amplitude gain is 1.48. The Rayleigh fading channel consists of 16 paths. The operating carrier frequency is 1.9 GHz and the vehicle speed is 40 Km/hr, hence the Doppler frequency shift is 70.36 Hz. By transmitting X_0 and X_1 for M times, the proposed estimator makes its estimation based on Algorithm 1. This is equivalent to having an OFDM training symbol of size $N = 4M$. Table I shows the trade-off performance between estimation accuracy and spectral efficiency for different M values. Simulation is performed over an AWGN channel with average SNR = 5 dB and $\varepsilon = 0.3$ for 2000 independent trials in order to measure the mean square error (MSE) in the CFO estimation to assess the estimation accuracy. The training symbol size, in terms of number of SCs, is used as an indication of the spectral efficiency. The larger

[1]In Algorithm 1, X_0 and X_1 are transmitted alternately so as to compensate for the effect of accumulated CFO over multiple OFDM symbols, as in this case the accumulate effect is common in both the numerator and denominator.

[2]A data frame is a collection of OFDM symbols.

TABLE I
THE TRADE-OFF BETWEEN ESTIMATION ACCURACY AND SPECTRAL EFFICIENCY

Transmission Times (M)	4	8	16	32
Training Symbol Size (SCs)	32	64	128	256
MSE (%)	0.42	0.2	0.1	0.05

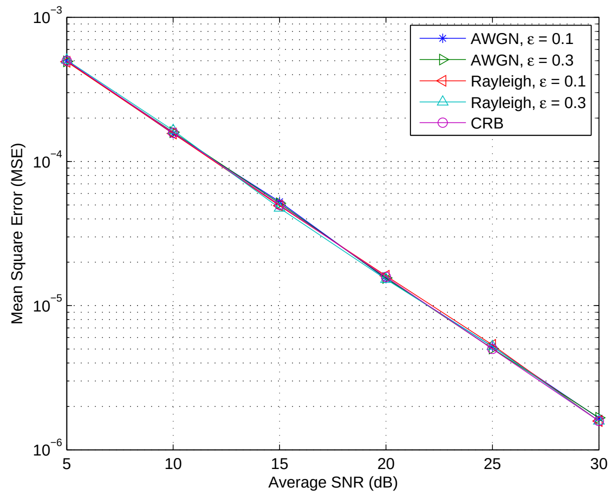

Fig. 1. Mean square error versus SNR over AWGN and Rayleigh fading channels.

the symbol size, the lower the spectral efficiency. It is observed that the estimation accuracy is improved with M, as this helps in removing the channel noise effect while performing the CFO estimation. However, this is achieved at the cost of a reduced spectral efficiency for the proposed estimator.

Figure 1 plots the performance of the proposed CFO estimator, with $M = 32$, over AWGN and Rayleigh fading channels for different SNR and CFO values. As shown, the MSE of the proposed estimator satisfies the Cramer-Rao bound for different CFO values, which means that the proposed estimator is an MLE.

Figure 2 shows the performance of the proposed estimator, with $M = 32$, in a Rayleigh fading channel for different CFO and SNR values. The estimator shows good performance over a wide range of CFO values, with an average estimation error of less than 2.38%, 1.49%, and 0.3% at SNR values of $5, 10$, and 15 dB, respectively.

VI. CONCLUSION

In this letter, a novel data-aided CFO estimator is proposed. The proposed estimator relies on transmitting the eigenvectors corresponding to a two-dimensional diagonal matrix to estimate the CFO value. The estimator exhibits high accuracy in CFO estimation over AWGN and Rayleigh fading channels, and is proven to be an MLE. In addition, the proposed estimator has low computational complexity, as it requires no complex operations as compared with existing CFO estimators.

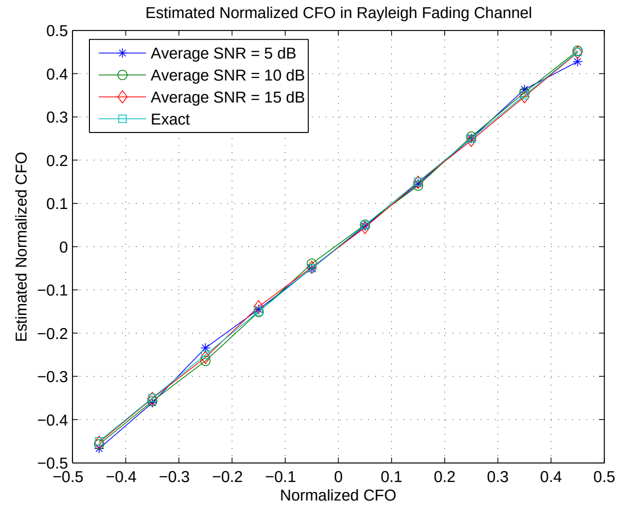

Fig. 2. CFO estimation versus actual CFO value over a Rayleigh fading channel.

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