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Robust Vaccination Strategy based on Dynamic Game for Uncertain SIR Time-Delay Model

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Abstract— **In this paper, a robust Pareto suboptimal strategy for an uncertain susceptible-infected-recovered (SIR) model with state delay is investigated, based on the static output feedback (SOF). After linearizing the original nonlinear SIR model, a sufficient condition for the existence of a proposed strategy set is derived in terms of high-order cross-coupled matrix equations (HCMEs). Using the guaranteed cost control technique, both robust stability and existence of the cost bound are attained. To avoid high complexity of directly solving the HCMEs, a recursive algorithm based on the linear matrix inequality (LMI) is presented. Finally, a practical SIR time-delay model is used to demonstrate the effectiveness and reliability of the proposed strategy.**

I. Introduction

Over the past few decades, the stability analysis and optimal control for the susceptible-infected-recovered (SIR) model have been investigated. In [1], to minimize the sizes of infected and susceptible populations, and to maximize the size of recovered population, a finite-time control problem was developed. In [2], impulse vaccination approach was investigated as a realistic vaccination strategy. In [3], stability analysis was performed on the SIR model with a generalized delay and the effects of vaccination and treatment. In [4], optimal vaccination strategy was proposed for the age-structured SIR model. In [5], the global stability of a delayed SIR epidemic model for an endemic equilibrium was analyzed. Subsequently, Elazzouzi et al. extended the study for local stability related to the disease-free equilibrium and global stability for the general SIR epidemic model with distributed delay that includes a general treatment function [6]. As another important extension, an SIR model with stochastic noise governed by the Itô differential equations was discussed, because the original SIR model can be captured as stochastic behaviors [7]. For optimal control, a two-point boundary value problem should be solved, which is difficult. Moreover, the highly nonlinear ordinary differential equation in the SIR model exacerbates the challenge.

On the other hand, if the SIR model is viewed as a large-scale dynamic system, many infection prevention and control stations can be considered for vaccination. Each control station can be regarded as a decision maker in a dynamic game. That is, the optimal control for the SIR model can be addressed using dynamic game theory. However, to the best of our knowledge, there are limited results for the vaccination strategy using the dynamic game theory.

In this study, a vaccination control problem for the SIR model with state delay is investigated. Because not all infection statuses can always be observed, a static output feedback (SOF) strategy is considered. In existing studies of the vaccination control problem based on the nonlinear time-delay SIR model, the linearization technique is used, and the resulting modeling error is represented as a norm-bounded uncertainty. As a result, it is shown that robust stability is attained because the guaranteed cost control technique [8] can be utilized, instead of the complicated two-point boundary value problem. Although the dynamic game problem for a time-delay system was investigated in [9], deterministic norm-bounded uncertainties caused by the linearization were not considered, except in our recent report [10]. Different from [10] in which no norm-bounded uncertainties in the input matrices are considered, in this work, we consider a more general model to capture these uncertainties for the first time. In addition, the sufficient condition for the existence of a proposed strategy set is established in terms of high-order cross-coupled matrix equations (HCMEs) such that the cost bound is minimized. To avoid high complexity in solving the HCMEs, a recursive algorithm based on the linear matrix inequality (LMI) is proposed. Moreover, the strong convergence that satisfies the norm convergence is proved. Finally, a practical SIR time-delay model is presented to demonstrate the effectiveness and reliability of the proposed strategy.

Notation: The notations used in this paper are fairly standard: I_n denotes the $n \times n$ identity matrix; $\| \cdot \|$ denotes the Euclidean norm of a matrix.

II. Motivation

In this section, we discuss in detail the suboptimal control problem for a class of nonlinear SIR time-delay systems, and provde the motivation for this study.

Let the sizes of susceptible, infected, and recovered populations at time t be $S(t)$, $I(t)$, and $R(t)$. Consider the SIR time-delay model defined by the following differential equations with N-multiple control stations as in [11]:

$$\dot{S}(t) = \mu - \sum_{k=1}^{N} v_k(t)S(t) - \mu S(t) - \beta S(t)I(t-h), \quad \text{(1a)}$$

$$\dot{I}(t) = \beta S(t)I(t-h) - (\gamma + \mu)I(t), \quad \text{(1b)}$$

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$$S(0) \geq 0,\ I(0) \geq 0. \tag{1c}$$

where In (1), $v_i(t)$, $i = 1, \ldots, N$, denotes the i-th vaccination rate, μ denotes the death rate that it is equal to the birth rate, β denotes the transmission rate, and γ denotes the recovery rate.

In [11], $v_i(t)$ is assumed to be constant because susceptible individuals are vaccinated with the rate constant. Furthermore, to the best of our knowledge, there are no multiple control stations in the existing results.

To minimize the sizes of infected and susceptible populations and to maximize the size of recovered population, the following cost function is defined for each control station:

$$L_i(v_i) = \int_0^{t_f} \Big\{ q_{i1}S(t) + q_{i2}I(t) + r_i v_i^2(t) \Big\} dt, \tag{2}$$

where q_{i1}, q_{i2}, and r_i, $i = 1, \ldots, N$, are given positive constant parameters, and t_f denotes the given termination time.

Without loss of generality, assume that all control stations can collaborate. In this case, the cost function in (2) can be centralized as follows:

$$L(v_1, \ldots, v_N) = \int_0^{t_f} \Big\{ \bar{q}_1 S(t) + \bar{q}_2 I(t) + G(t) \Big\} dt, \tag{3}$$

where

$$\bar{q}_1 := \frac{1}{N}\sum_{k=1}^{N} q_{1k},\ \bar{q}_2 := \frac{1}{N}\sum_{k=1}^{N} q_{2k},\ G(t) := \frac{1}{2N}\sum_{k=1}^{N} r_k v_k^2(t).$$

To establish the necessary condition for the suboptimal strategy set based on the linear quadratic control, Pontryagin's maximum principal for the time delay case [12] is applied. First, the following Hamiltonian $\mathscr{H}$ is defined:

$$\begin{aligned}\mathscr{H} &= \bar{q}_1 S(t) + \bar{q}_2 I(t) + G(t) \\ &\quad + \lambda_1(t)\left[\mu - \sum_{k=1}^{N} v_k(t)S(t) - \mu S(t) - \beta S(t) I(t-h)\right] \\ &\quad + \lambda_2(t)\Big[\beta S(t) I(t-h) - (\gamma+\mu) I(t)\Big].\end{aligned} \tag{4}$$

Hence, the use of maximum principal for the time delay case [12] results in the following:

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial \mathscr{H}}{\partial S(t)} - \chi_{[0,t_f-h]}(t)\frac{\partial \mathscr{H}}{\partial S(t-h)} \\ &= -\bar{q}_1 + \lambda_1(t)\left[\sum_{k=1}^{N} v_k(t) + \mu + \beta I(t-h)\right] \\ &\quad - \lambda_2(t)\beta I(t-h),\end{aligned} \tag{5a}$$

$$\begin{aligned}\dot{\lambda}_2 &= -\frac{\partial \mathscr{H}}{\partial I(t)} - \chi_{[0,t_f-h]}(t)\frac{\partial \mathscr{H}}{\partial I(t-h)} \\ &= -\bar{q}_2 + \lambda_2(t)(\gamma+\mu) \\ &\quad + \chi_{[0,t_f-h]}(t)\beta S(t)\Big(\lambda_1(t+h) - \lambda_2(t+h)\Big),\end{aligned} \tag{5b}$$

$$\frac{\partial \mathscr{H}}{\partial v_i(t)} = \frac{r_i}{N} v_i - \lambda_1(t) S(t) = 0,\ i = 1, \ldots, N, \tag{5c}$$

where

$$\chi_{[a,b]}(t) = \begin{cases} 1, & t \in [a,b] \\ 0, & \text{otherwise} \end{cases},\ \lambda_1(t_f) = \lambda_2(t_f) = 0.$$

Finally, the following two-point boundary value problem can be obtained:

$$\begin{aligned}\dot{S}^\dagger(t) &= \mu - \sum_{k=1}^{N}\frac{N}{r_k}\lambda_1^\dagger(t)[S^\dagger(t)]^2 - \mu S^\dagger(t) \\ &\quad - \beta S^\dagger(t) I^\dagger(t-h),\end{aligned} \tag{6a}$$

$$\dot{I}^\dagger(t) = \beta S^\dagger(t) I^\dagger(t-h) - (\gamma+\mu) I^\dagger(t), \tag{6b}$$

$$\begin{aligned}\dot{\lambda}_1^\dagger &= -\bar{q}_1 + \lambda_1^\dagger(t)\left[\sum_{k=1}^{N}\frac{N}{r_k}\lambda_1^\dagger(t) S^\dagger(t) + \mu + \beta I^\dagger(t-h)\right] \\ &\quad - \lambda_2^\dagger(t)\beta I^\dagger(t-h),\end{aligned} \tag{6c}$$

$$\begin{aligned}\dot{\lambda}_2^\dagger &= -\bar{q}_2 + \lambda_2^\dagger(t)(\gamma+\mu) \\ &\quad + \chi_{[0,t_f-h]}(t)\beta S^\dagger(t)\Big(\lambda_1^\dagger(t+h) - \lambda_2^\dagger(t+h)\Big),\end{aligned} \tag{6d}$$

$$S^\dagger(0) \geq 0,\ I^\dagger(0) \geq 0,\ \lambda_1^\dagger(t_f) = \lambda_2^\dagger(t_f) = 0. \tag{6e}$$

By computing these equations numerically, the following vaccination strategy can be designed:

$$v_i(t) = \frac{N}{r_i}\lambda_1(t) S^\dagger(t),\ i = 1, \ldots, N. \tag{7}$$

It is well known that the obtained two-point boundary value problem (6) is highly complex to solve, even though a numerical algorithm is applied. This is because it contains a time delay and many decision makers act as control stations. In contrast, when applying strategy set (7), the number of susceptible people, $S(t)$, is required. However, it is difficult to obtain this number in reality. Our goal here is to overcome these drawbacks by applying a linear SOF vaccination strategy with a dynamic game for a linear uncertain time-delay system, because $I(t)$ can be more easily and accurately observed than $S(t)$.

III. Problem Formulation

Consider the following linearized uncertain time-delay model:

$$\begin{aligned}\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_h + \Delta A_h(t)]x(t-h) \\ &\quad + \sum_{k=1}^{N}[B_k + \Delta B_k(t)]u_k(t),\end{aligned} \tag{8a}$$

$$x(t) = \phi(t),\ -h \leq t \leq 0, \tag{8b}$$

$$y_i(t) = C_i x(t),\ i = 1, 2 \ldots, N, \tag{8c}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector; $u_i(t) \in \mathbb{R}^m$, $i = 1, \ldots, N$, denotes the i-th control input; $y_i(t) \in \mathbb{R}^{n_y}$, $i = 1, \ldots, N$, denotes the i-th measurement output; $\phi(t) \in \mathbb{R}^n$ denotes the initial function for $x(t)$; and $h \in (0, \infty)$ denotes the time delay. Coefficient matrices A, A_h, B_i, and C_i, $i = 1, \ldots, N$, are known to be constant with appropriate dimensions; $\Delta A(t)$, $\Delta A_h(t)$ and $\Delta B_i(t)$, $i = 1, \ldots, N$, are the time-varying matrix functions representing the uncertainties in matrices A, A_h and B_i, $i = 1, \ldots, N$.

It is assumed that the uncertainties can be described by

$$\left[\Delta A(t)\ \Delta A_h(t)\ \Delta B_i(t)\right] = H\Delta(t)\left[E_a\ E_h\ E_{bi}\right], \quad (9)$$

where $\Delta(t) \in \mathbb{R}^{p\times q}$ is an unknown real time-varying matrix with Lebesgue measurable functions bounded by

$$\Delta^T(t)\Delta(t) \le I_q,\ \forall t \in [-h, \infty). \quad (10)$$

In the presence of multiple decision makers, the following weighted sum of the cost function is introduced [13]:

$$J(u_1, \dots, u_N, \phi) = \sum_{k=1}^{N} \rho_k J_k(u_k, \phi), \quad (11)$$

where

$$0 < \rho_i < 1\ ,\ \sum_{k=1}^{N} \rho_k = 1,$$

$$J_i(u_i) = \int_0^\infty \Big\{x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t)\Big\}dt.$$

Furthermore, without loss of generality, $Q_i = Q_i^T > 0$ and $R_i = R_i^T > 0$, $i = 1, \dots, N$, are assumed.

As only partial information is available, an SOF strategy is considered in the following form:

$$u_i(t) = F_i y_i(t) = F_i C_i x(t). \quad (12)$$

Our problem under consideration is as follows:
Find the SOF Pareto strategy in (12)*, such that the robust stability is guaranteed and the bound of the quadratic cost function in* (11) *is minimized.*

Next, the robust SOF Pareto strategy is derived by means of the LMI conditions.

IV. Main Result

To simplify derivation, following assumptions are made:

$$\int_{-h}^{0} \phi(s)\phi^T(s)ds = U,\ \mathbb{E}[x(0)x^T(0)] = M.$$

It should be noted the matrices U and M are constant values.

Theorem 1: Consider the linearized uncertain time-delay model in (8) with multiple decision makers $u_i(t)$. Suppose there exists a matrix solution set for the real symmetric matrices $P > 0$, $W > 0$ and $S > 0$, an SOF feedback strategy set F_i, $i = 1, \dots, N$, and positive parameters ε_1 and ε_2, such that the following HCMEs are satisfied:

$$\Phi_1(P, S, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2) = 0, \quad (13a)$$

$$\Phi_2(P, S, W, \varepsilon_2) = 0, \quad (13b)$$

$$\Phi_3(P, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2) = 0, \quad (13c)$$

$$\Phi_4^i(P, S, F_1, \dots, F_N, \varepsilon_1) = 0,\ i = 1, \dots, N, \quad (13d)$$

$$\varepsilon_1 = \sqrt{\frac{\mathbf{Tr}\left[H^T PSPH\right]}{\mathbf{Tr}\left[E_F S E_F^T\right]}}, \quad (13e)$$

$$\varepsilon_2 = \sqrt{\frac{\mathbf{Tr}\left[H^T PSPH\right]}{\mathbf{Tr}\left[E_h \mathcal{E}(W)^{-1} A_h^T PSPA_h \mathcal{E}(W)^{-1} E_h^T\right]}}, \quad (13f)$$

$$W > \varepsilon_2 E_h^T E_h,\ C_i S C_i^T > 0, \quad (13g)$$

where

$$\begin{aligned}
&\Phi_1(P, S, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2)\\
&\quad = M + SA_F^T + A_F S\\
&\qquad + (\varepsilon_1^{-1} + \varepsilon_2^{-1})(SPHH^T + HH^T PS)\\
&\qquad + SPA_h\mathcal{E}(W)^{-1}A_h^T + A_h\mathcal{E}(W)^{-1}A_h^T PS,\\
&\Phi_2(P, S, W, \varepsilon_2)\\
&\quad = U + S - \mathcal{E}(W)^{-1}A_h^T PSPA_h\mathcal{E}(W)^{-1},\\
&\Phi_3(P, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2)\\
&\quad = PA_F + A_F^T P + (\varepsilon_1^{-1} + \varepsilon_2^{-1})PHH^T P + W\\
&\qquad + \varepsilon_1 E_F E_F^T + \sum_{k=1}^{N} \rho_i \Big(Q_k + C_k^T F_k^T R_k F_k C_k\Big)\\
&\qquad + PA_h\mathcal{E}(W)^{-1}A_h^T P,\\
&\Phi_4^i(P, S, F_1, \dots, F_N, \varepsilon_1)\\
&\quad = \Big(B_i^T P + \rho_i R_i F_i C_i + \varepsilon_1 E_{bi}^T E_F\Big)SC_i^T,
\end{aligned}$$

$$A_F := A + \sum_{k=1}^{N} B_k F_k C_k,\ E_F := E_a + \sum_{k=1}^{N} E_{bk} F_k C_k,$$

$$\mathcal{E}(W) := W - \varepsilon_2 E_h^T E_h.$$

Then, robust SOF Pareto suboptimal strategy set is given by

$$u_i^*(t) = F_i^* y_i(t) = F_i^* C_i x(t). \quad (14)$$

Furthermore, the following inequality holds:

$$J(u_1, \dots, u_N, \phi(0)) \le \mathbf{Tr}[MP] + \mathbf{Tr}[UW]. \quad (15)$$

Proof: First, the robust stability strategy and the existence of cost bound under the condition of multiple decision makers are proved. Consider the following parameter-independent Lyapunov function candidate:

$$V(x(t), t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Wx(s)ds. \quad (16)$$

Using the following well-known results:

$$\begin{aligned}
&x^T(t)\Big(PH\Delta(t)E_F + E_F^T\Delta^T(t)H^T P\Big)x(t)\\
&\quad \le x^T(t)\Big(\varepsilon_1^{-1}PHH^T P + \varepsilon_1 E_F^T E_F\Big)x(t),\ \varepsilon_1 > 0,\\
&2x^T(t)PH\Delta(t)E_h x(t-h)\\
&\quad \le \varepsilon_1^{-1}x^T(t)PHH^T Px(t)\\
&\qquad + \varepsilon_1 x^T(t-h)E_h^T E_h x(t-h),\ \varepsilon_2 > 0,
\end{aligned}$$

time derivative of (16) along with closed-loop model subject to the SOF strategy (12) with respect to time t is given by

$$\begin{aligned}
&\dot{V}(x(t), t) + x^T(t)\left[\sum_{k=1}^{N} \rho_k(Q_k + C_k^T F_k^T R_k F_k C_k)\right]x(t)\\
&\quad = x^T(t)P\dot{x}(t) + x^T(t)P\dot{x}(t)\\
&\qquad + x^T(t)Wx(t) - x^T(t-h)Wx(t-h)\\
&\quad \le \xi^T(t)\Sigma\xi(t),
\end{aligned} \quad (17)$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix},\ \Sigma = \begin{bmatrix} \Sigma_{11} & PA_h \\ A_h^T P & \Sigma_{22} \end{bmatrix},$$

$$\Sigma_{11} = PA_F + A_F^T P + (\varepsilon_1^{-1} + \varepsilon_2^{-1}) PHH^T P + W + \varepsilon_1 E_F^T E_F + \sum_{k=1}^{N} \rho_k \Big(Q_k + C_k^T F_k^T R_k F_k C_k \Big),$$
$$\Sigma_{22} = -W + \varepsilon_2 E_h^T E_h.$$

Therefore, if inequality $\Sigma < 0$ holds, the original linearized uncertain time-delay model in (8) is stable against the uncertainty $\Delta(t)$. In this case, integrating both sides of (17) yields the following inequality:

$$J(u_1, \dots, u_N, \phi(0)) < x^T(0) P x(0) + \int_{-h}^{0} \phi^T(s) W \phi(s) ds = \mathbf{Tr}[MP] + \mathbf{Tr}[UW]. \quad (18)$$

Second, the following optimization problem related to the cost bound is investigated:

$$\min_{P, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2} \Big(\mathbf{Tr}[MP] + \mathbf{Tr}[UW] \Big) \text{ s.t. } \Sigma < 0. \quad (19)$$

Note that this problem is purely an optimization problem with the constraint of bilinear matrix inequality (BMI). Therefore, it is difficult to solve in general because the BMI optimization problem is close to NP-hard. Hence, the necessary condition is established by using the Karush-Kuhn-Tucker (KKT) conditions instead of directly solving the BMI optimization problem.

The Lagrangian, $\boldsymbol{L}$, is defined as

$$\boldsymbol{L} = \mathbf{Tr}[MP] + \mathbf{Tr}[UW] + \mathbf{Tr}[T\Sigma_0], \quad (20)$$

where T is the Lagrange multiplier and

$$\Sigma_0 := \Sigma_{11} - PA_h \Sigma_{22}^{-1} A_h^T P.$$

Inequality $\Sigma_0 < 0$ is equivalent to $\Sigma < 0$, because of Schur complement formula. Partial derivatives of $\boldsymbol{L}$ are given by

$$\frac{\partial \boldsymbol{L}}{\partial P} = \Phi_1(P, T, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2) = 0, \quad (21a)$$
$$\frac{\partial \boldsymbol{L}}{\partial W} = \Phi_2(P, T, W, \varepsilon_2) = 0, \quad (21b)$$
$$\frac{\partial \boldsymbol{L}}{\partial T} = \Phi_3(P, W, F_1, \dots, F_N, \varepsilon_1, \varepsilon_2) = 0, \quad (21c)$$
$$\frac{\partial \boldsymbol{L}}{\partial F_i} = \Phi_4^i(P, T, F_1, \dots, F_N, \varepsilon_1) = 0, \quad (21d)$$
$$\frac{\partial \boldsymbol{L}}{\partial \varepsilon_1} = \Phi_5(P, T, F_1, \dots, F_N, \varepsilon_1) = 0, \quad (21e)$$
$$\frac{\partial \boldsymbol{L}}{\partial \varepsilon_2} = \Phi_6(P, T, W, F_1, \dots, F_N, \varepsilon_2) = 0, \quad (21f)$$

where

$$\Phi_5(P, T, F_1, \dots, F_N, \varepsilon_1) = -\varepsilon_1^{-2} \mathbf{Tr}\Big[H^T PTPH\Big] + \mathbf{Tr}\Big[E_F T E_F^T\Big],$$
$$\Phi_6(P, T, W, F_1, \dots, F_N, \varepsilon_2) = -\varepsilon_2^{-2} \mathbf{Tr}\Big[H^T PTPH\Big] + \mathbf{Tr}\Big[E_h \mathcal{E}(W)^{-1} A_h^T PTPA_h \mathcal{E}(W)^{-1} E_h^T\Big].$$

From (21a) to (21d), algebraic matrix equations (13a) and (13d) can be obtained. Moreover, solving (21e) and (21f), parameters ε_1 and ε_2 as given in (13e) and (13f) can be derived. Therefore, if conditions (21) are established, the cost bound of $\mathbf{Tr}[MP] + \mathbf{Tr}[UW]$ attains a minimum value. ■

V. Numerical Algorithm

To determine robust vaccination Pareto suboptimal strategy set in (14), HCMEs (13) should be solved. Newton's method can be applied to the HCMEs, which attains the local quadratic convergence if initial conditions are close to required solution set. However, Newton's method needs to compute Jacobian of HCMEs, which is highly complicated and requires tedious algebra. We propose novel numerical algorithm based on LMI and prove its strong convergence. First, recursive algorithm is established as follows:

Step 1. *Set the initial values for each $F_i^{(0)}$, $i = 1, \dots, N$, such that the closed-loop linearized uncertain time-delay model is stable;*

Step 2. *Solve the following optimization problem with respect to $P^{(\ell)}$, $W^{(\ell)}$, $\varepsilon_1^{(\ell)}$ and $\varepsilon_2^{(\ell)}$:*

$$\min_{P^{(\ell)}, W^{(\ell)}, \varepsilon_1^{(\ell)}, \varepsilon_2^{(\ell)}} \Big(\mathbf{Tr}[MP^{(\ell)}] + \mathbf{Tr}[UW^{(\ell)}] \Big),$$
$$\text{s.t.} \begin{bmatrix} \Gamma_{11}^{(\ell)} & P^{(\ell)} H & P^{(\ell)} H & P^{(\ell)} A_h \\ H^T P^{(\ell)} & -\varepsilon_1^{(\ell)} I_p & 0 & 0 \\ H^T P^{(\ell)} & 0 & -\varepsilon_2^{(\ell)} I_p & 0 \\ A_h^T P^{(\ell)} & 0 & 0 & \Gamma_{22}^{(\ell)} \end{bmatrix} < 0, \quad (22)$$

where

$$\Gamma_{11}^{(\ell)} = P^{(\ell)} A_F^{(\ell-1)} + A_F^{(\ell-1)T} P^{(\ell)} + \varepsilon_1^{(\ell)} E_F^{(\ell)T} E_F^{(\ell)} + W^{(\ell)} + \sum_{k=1}^{N} \rho_k \Big(Q_k + C_k^T F_k^{(\ell)T} R_k F_k^{(\ell)} C_k \Big),$$
$$A_F^{(\ell)} := A + \sum_{k=1}^{N} B_k F_k^{(\ell)} C_k, \ E_F^{(\ell)} := E_a + \sum_{k=1}^{N} E_{bk} F_k^{(\ell)} C_k,$$
$$\Gamma_{22}^{(\ell)} := -W^{(\ell)} + \varepsilon_2^{(\ell)} E_h^T E_h;$$

Step 3. *Solve the following Lyapunov equation for $T^{(\ell)}$:*

$$\Phi_1(P^{(\ell)}, T^{(\ell)}, W^{(\ell)}, F_1^{(\ell-1)}, \dots, F_N^{(\ell-1)}, \varepsilon_1^{(\ell)}, \varepsilon_2^{(\ell)}) = 0; \quad (23)$$

Step 4. *Solve the SOF strategy, $F_i^{(\ell)}$, in terms of the following linear algebraic matrix equation:*

$$\Phi_4^i(P^{(\ell)}, T^{(\ell)}, F_1^{(\ell)}, \dots, F_N^{(\ell)}, \varepsilon_1^{(\ell)}) = 0, \ i = 1, \dots, N; \quad (24)$$

Step 5. *For any appropriate value of $\boldsymbol{z}$, set*

$$\boldsymbol{z}^{(\ell+1)} = \frac{1}{\ell} \boldsymbol{z} + \left(1 - \frac{1}{\ell}\right) \mathbb{T}(\boldsymbol{z}^{(\ell)}), \quad (25)$$

where

$$\boldsymbol{z}^{(\ell)} = \mathbb{T}(\boldsymbol{z}^{(\ell-1)}),$$
$$\boldsymbol{z}^{(\ell)} := \big[\ \mathrm{vec}[P^{(\ell)}] \ \ \mathrm{vec}[T^{(\ell)}] \ \ \mathrm{vec}[W^{(\ell)}] \ \ \mathrm{vec}[F_1^{(\ell)}] \ \cdots \ \mathrm{vec}[F_N^{(\ell)}] \ \ \varepsilon_1^{(\ell)} \ \ \varepsilon_2^{(\ell)} \ \big];$$

Step 6. *If the following inequality is satisfied for a given sufficiently small δ during the iterations of Steps 2 to 5, the iteration stop and $F_i^{(\ell)}$ $i = 1, \ldots, N$, are obtained as the feedback gain F_i $i = 1, \ldots, N$:*

$$\|\mathbb{F}(P^{(\ell)}, T^{(\ell)}, W^{(\ell)}, F_1^{(\ell)}, \ldots, F_N^{(\ell)}, \varepsilon_1^{(\ell)}, \varepsilon_2^{(\ell)})\| < \delta, \quad (26)$$

where

$$\mathbb{F}(P, T, W, F_1, \ldots, F_N, \varepsilon_1, \varepsilon_2) = \begin{bmatrix} \Phi_1(P, T, W, F_1, \ldots, F_N, \varepsilon_1, \varepsilon_2) \\ \Phi_2(P, T, W, \varepsilon_1) \\ \Phi_3(P, W, F_1, \ldots, F_N, \varepsilon_1, \varepsilon_2) \\ \Phi_4^1(P, T, F_1, \ldots, F_N, \varepsilon_1) \\ \vdots \\ \Phi_4^N(P, T, F_1, \ldots, F_N, \varepsilon_1) \\ \Phi_5(P, T, F_1, \ldots, F_N, \varepsilon_1) \\ \Phi_6(P, T, W, F_1, \ldots, F_N, \varepsilon_2) \end{bmatrix};$$

Otherwise, if the number of iterations reaches a preset threshold, it is declared that there is no strategy set, and the algorithm stops.

Theorem 2: If $\mathbb{T}$ is a monotone nonexpansive mapping with fixed point, then $\{\boldsymbol{z}^{(\ell)}\}$ converges weakly to fixed point.

Proof: Based on the existing result in [14]. In Lemma 1 of Appendix, we define the following sequence in (25):

$$\boldsymbol{z}^{(\ell+1)} = \alpha_\ell \boldsymbol{z} + (1 - \alpha_\ell)\, \mathbb{T}(\boldsymbol{z}^{(\ell)}) \quad (27)$$

where for any $\boldsymbol{z}$,

$$\alpha_\ell = \frac{1}{\ell}, \ \ell = 1, 2, \ldots.$$

It is easy to show that

$$0 < \alpha_\ell = \frac{1}{\ell} < 1, \quad (28a)$$

$$\lim_{\ell \to \infty} \alpha_\ell = \lim_{\ell \to \infty} \frac{1}{\ell} = 0, \quad (28b)$$

$$\sum_{\ell=1}^{\infty} \alpha_\ell = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = \infty, \quad (28c)$$

$$\sum_{\ell=1}^{\infty} |\alpha_{\ell+1} - \alpha_\ell| = \sum_{\ell=1}^{\infty} \left(\frac{1}{\ell} - \frac{1}{\ell+1}\right) = 1 < \infty. \quad (28d)$$

Therefore, the conditions in Lemma 1 are satisfied and thus the resulting sequence in (25) is convergent. ∎

VI. Uncertain SIR Model

In this section, the control strategy for a practical uncertain SIR model is derived. Consider the SIR model defined by the following differential equations:

$$\dot{S}(t) = \left(1 - \sum_{k=1}^{N} v_k(t)\right) b - \mu S(t) - \beta S(t) I(t-h), \quad (29a)$$

$$\dot{I}(t) = \beta S(t) I(t-h) - (\gamma + \mu) I(t), \quad (29b)$$

$$\dot{R}(t) = b \sum_{k=1}^{N} v_k(t) - \mu R(t) + \gamma I(t), \quad (29c)$$

$$S(0) \geq 0, \ I(0) \geq 0, \ R(0) = 0, \quad (29d)$$

where b denotes the ratio of the newborn population. Other parameters are defined by the SIR model in (1). Only a two-dimensional dynamic system needs to be considered under the following assumptions:

$$N(t) = S(t) + I(t) + R(t) = \frac{b}{\mu}. \quad (30)$$

Then, consider the following linearized SIR time-delay model:

$$\dot{\bar{S}}(t) = (-\mu - \beta I^*)\bar{S}(t) - \beta S^* \bar{I}(t-h) - b \sum_{k=1}^{N} v_k(t) + o\Big(\sqrt{\bar{S}^2 + \bar{I}^2}^2\Big), \quad (31a)$$

$$\dot{\bar{I}}(t) = \beta I^* \bar{S}(t) - (\gamma + \mu)\bar{I}(t) + \beta S^* \bar{I}(t-h) + o\Big(\sqrt{\bar{S}^2 + \bar{I}^2}^2\Big), \quad (31b)$$

where

$$\bar{S}(t) = S(t) - S^*, \ \bar{I}(t) = I(t) - I^*,$$

$$\lim_{(\bar{S}, \bar{I}) \to (+0, +0)} \frac{o\Big(\sqrt{\bar{S}^2 + \bar{I}^2}^2\Big)}{\sqrt{\bar{S}^2 + \bar{I}^2}} = 0,$$

with S^* and I^* denoting the equilibrium solutions of $S(t)$ and $I(t)$, respectively.

For the SIR time-delay model in (30), the following parameters are set:

$$\beta = 0.205, \ \gamma = 0.08, \ \mu = 0.1, \ b = 0.1,$$
$$S(t) = 0.75, \ I(t) = 0.25, \ -5 \leq t \leq 0.$$

Finally, $o\Big(\sqrt{\bar{S}^2 + \bar{I}^2}^2\Big)$ is modeling error, which can be regarded as having deterministic uncertainties. As a result, following uncertain SIR time-delay model can be obtained:

$$\dot{x}(t) = [A + H\Delta(t)E_a]x(t) + [A_h + H\Delta(t)E_h]x(t-h) + \sum_{k=1}^{3}[B_k + H\Delta(t)E_{bk}]u_k(t), \quad (32a)$$

$$x(t) = \begin{bmatrix} \bar{S}(t) \\ \bar{I}(t) \end{bmatrix}, \ \phi(t) = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}, \ -5 \leq t \leq 0, \quad (32b)$$

$$y_i(t) = C_i x(t), \ i = 1, 2, 3. \quad (32c)$$

where

$$v_i(t) = u_i(t) = F_i \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) = F_i\big(I(t) - I^*\big), \ i = 1, 2, 3,$$

$$A = \begin{bmatrix} -\mu - \beta I^* & 0 \\ \beta I^* & -(\gamma + \mu) \end{bmatrix} = \begin{bmatrix} -1.1389 \times 10^{-1} & 0 \\ 1.3889 \times 10^{-2} & -1.8000 \times 10^{-1} \end{bmatrix},$$

$$A_h = \begin{bmatrix} 0 & -\beta S^* \\ 0 & \beta S^* \end{bmatrix} = \begin{bmatrix} 0 & -1.8000 \times 10^{-1} \\ 0 & 1.8000 \times 10^{-1} \end{bmatrix},$$

$$B_1 = B_2 = B_3 = \begin{bmatrix} -b \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \ C_i = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$H = 0.001 \times \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \ E_a = \begin{bmatrix} 0.1 & 3 \end{bmatrix}, E_h = \begin{bmatrix} 0.01 & 10 \end{bmatrix},$$

$$E_{b1} = E_{b2} = E_{b3} = 0.1.$$

VII. Numerical Example

To demonstrate the effectiveness of the proposed strategy set and the iterative algorithm, a numerical example based on the practical SIR model is studied.

First, HCMEs (13) are solved using the proposed LMI-based algorithm in (25). The computed strategy set in (14) and with the related solution matrices are given by

$$P = \begin{bmatrix} 5.0642 & 4.0057 \\ 4.0057 & 8.7980 \times 10 \end{bmatrix},$$
$$S = \begin{bmatrix} 9.5876 & -4.6047 \\ -4.6047 & 4.8868 \end{bmatrix},$$
$$W = \begin{bmatrix} 1.6693 \times 10^{-5} & 9.2382 \times 10^{-5} \\ 9.2382 \times 10^{-5} & 1.4922 \times 10 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} -2.7717 \times 10^{-2} \end{bmatrix},$$
$$F_2 = \begin{bmatrix} -3.6950 \times 10^{-2} \end{bmatrix},$$
$$F_3 = \begin{bmatrix} -3.6950 \times 10^{-2} \end{bmatrix},$$
$$\varepsilon_1 = 3.3029 \times 10^{-3},\ \varepsilon_2 = 9.4043 \times 10^{-4}.$$

Next, trajectories of uncertain SIR time-delay model are confirmed. It should be noted that SOF means static out feedback strategies are applied. In addition, compared with the proposed strategy set, the feed-forward strategy is tested. In this case, $v_i(t) = 0.01$, $i = 1, 2, 3$, is time invariant.

Fig. 1 shows how the number of infected people changes over time. It is observed that the number is reduced by vaccination strategy based on the Pareto suboptimal. Therefore, the effectiveness of proposed vaccination strategy is demonstrated.

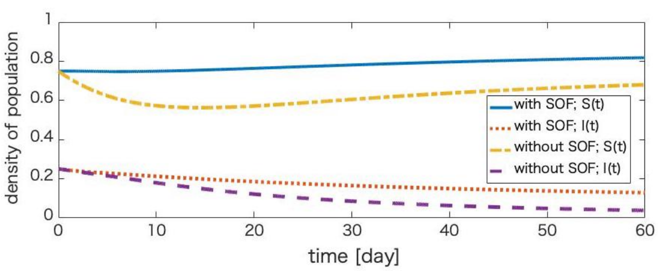

Fig. 1. Trajectories of SIR time-delay model

VIII. Conclusions

In this paper, we have proposed a robust vaccination Pareto suboptimal strategy for the SIR time-delay model in terms of the SOF strategy. Using the linearized SIR model, a sufficient condition for the existence of the Pareto suboptimal strategy set has been established. It has been shown that the proposed strategy set can be obtained by solving HCMEs. Subsequently, to solve HCMEs, a new recursive algorithm based on the LMI has been established. Finally, the proposed strategy set is applied to a practical nonlinear SIR time-delay model. The results of a case study demonstrated that the number of infected people is reduced by the proposed vaccination strategy.

Appendix

The following lemma [14] plays an important role in proving Theorem 2.

Lemma 1: Let α_n be a sequence in $[0, 1]$ such that

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \sum_{n=1}^{\infty} |\alpha_{n+1} - a_n| < \infty.$$

Let K be a Hilbert space H and $\mathbb{T}$ be a nonexpansive mapping on K with a nonempty fixed point set. Then, for any $x \in K$, the sequence

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\mathbb{T}(x_n),\ n = 0, 1, \ldots,$$

strongly converges to the fixed point. It is called the norm convergent.

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