Stochastic Modeling of Single-Hop Cluster Stability in Vehicular Ad Hoc Networks

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Abstract—Node clustering is a potential approach to improve the scalability of networking protocols in vehicular ad hoc networks (VANETs). High relative vehicle mobility and frequent network topology changes inflict new challenges on maintaining stable clusters. As a result, cluster stability is a crucial measure of the efficiency of clustering algorithms for VANETs. This paper presents a stochastic analysis of the vehicle mobility impact on single-hop cluster stability. A stochastic mobility model is adopted to capture the time variations of intervehicle distances (distance headways). Firstly, we propose a discrete-time lumped Markov chain to model the time variations of a system of distance headways. Secondly, the first passage time analysis is used to derive probability distributions of the time periods of invariant cluster-overlap state and cluster-membership as measures of cluster stability. Thirdly, queueing theory is utilized to model the limiting behaviors of the numbers of common and unclustered nodes between neighboring clusters. Numerical results are presented to evaluate the proposed models, which demonstrate a close agreement between analytical and simulation results.

I. INTRODUCTION

A vehicular ad hoc network (VANET) is a promising addition to our future intelligent transportation systems, which is provisioned to support various safety and infotainment applications [2], [3]. Urban roads and highways are highly susceptible to a large number of vehicles and traffic jams. Therefore, networking protocols for VANETs should be scalable to support such large scale networks. Node clustering is a network management strategy in which nearby nodes are grouped into a set called a cluster. In each cluster, a node, called cluster head (CH), is elected to manage the cluster. The remaining nodes are called cluster members (CMs), each belonging to one or multiple clusters.

Node clustering, just as in traditional ad hoc networks, is a potential approach to improve the scalability of networking protocols such as for routing and medium access control in VANETs. For medium access control protocols, the CH can act as a central coordinator that manages the access of its CMs to the wireless channel(s) [4]. For routing protocols, CHs can be made responsible for the discovery and maintenance of routing paths, thus limiting the control-message overhead in these processes [5]. Despite the potential benefits of node clustering, forming and maintaining the clusters require explicit exchange of control messages. In VANETs, vehicles move with high and variable speeds, causing frequent changes in the network topology, which can significantly increase the cluster maintenance cost. Therefore, how to form stable clusters that last for a long time is a major issue in node clustering of VANETs.

In a highly dynamic VANET, vehicles join and leave clusters along their travel route, resulting in changes in cluster structure. The temporal changes in cluster structure are either internal or external [6]. An internal change in the cluster structure is concerned with a change inside the cluster such as when vehicles join or leave the cluster, resulting in a change in cluster-membership. Frequent changes in the internal cluster structure consume network radio resources and cause service disruption for the cluster-based network protocols (e.g., in intracluster resource allocation, route discovery, and message delivery). Therefore, analyzing the impact of vehicle mobility on the rate at which nodes enter and leave a cluster is an important measure of internal cluster stability. This metric has been adopted by researchers to evaluate the performance of their proposed clustering algorithms through simulations [7]–[9]. A higher rate of cluster-membership changes indicates a smaller time period of invariant cluster-membership and, therefore, lower internal cluster stability.

On the other hand, an external change in the cluster structure is concerned with the relationship of a cluster with other clusters in a network. One metric that evaluates the external relationship of a cluster is its overlapping ranges with neighboring clusters. The time variations of the distance between two neighboring CHs, due to vehicle mobility, can cause the coverage ranges of the clusters to overlap. As the overlapping range between two clusters increases, the two clusters may merge into a single cluster [4], [7], [10]. Frequent splitting and merging of clusters increase the control overhead and drain the radio resources [9], [11], [12]. In general, a non-overlapping clustered structure produces a less number of clusters and lowers the design complexity of network protocols that run on the clusters. For example, two clusters may utilize the same radio resources at the same time if they are non-neighboring clusters [13] [14]. On the other hand, a highly overlapping clustered structure may cause complexity in the channel assignment, lead to broadcast storms, and form long hierarchical routes. Additional radio resources ought to be used to prevent inter-cluster interference due to overlapping, for example, assigning different time frames for neighboring clusters [15] and assigning different transmission codes to CMs located in a possibly overlapping region [16]. Although researchers have favored forming non-overlapping (or reduced overlapping) clustered structure [12] [11] [10] [16], encountering overlapping clusters during the network runtime is inevitable, especially in a highly mobile network. Overlapping clusters have received significant attention since the work by Palla et al. [17]. It is shown that real networks are better characterized by well-defined statistics of overlapping and nested clusters rather than disjoint clusters. Additionally, overlapping structure can provide a ground for cooperation among the overlapping clusters. For example, in [18], overlapping is used for cooperative interference management for small cell networks. Regardless of whether or not cluster overlapping is preferred, characterizing the overlapping state between neighboring clusters and its changes over time becomes crucial in the presence of node mobility. A higher rate

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of cluster-overlap state change indicates a shorter time period of unchanged cluster-overlap state and, therefore, lower external cluster stability.

Despite the importance of cluster stability as a measure of clustering algorithm efficiency in VANETs, characterizing cluster stability has taken the form of simulations [7]–[9] or case studies [19] in the literature.

In this paper, we present a stochastic analysis of two cluster stability metrics: the change rate in the overlap state between neighboring clusters as a measure of external cluster stability and the change rate in cluster-membership as a measure of internal cluster stability. We adopt a stochastic vehicle mobility model that describes the time variations of intervehicle distances and accounts for the realistic dependency of these variations at consecutive time steps. Firstly, the distance between two vehicles, separated by a number of vehicles on a highway, is modeled as a discrete-time Markov chain with a reduced dimensionality. Using the first passage time analysis, we derive the probability distributions of the time before the first change in the cluster-overlap state and the time interval between two successive changes in cluster-overlap state of two neighboring clusters. Secondly, the distributions of the time before the first cluster-membership change and the time interval between two successive cluster-membership changes are derived. Thirdly, the overlapping region between overlapping clusters and the unclustered region between non-overlapping clusters are modeled as a storage buffer in a two-state random environment. Using G/G/1 queuing theory, the steady-state distributions of the numbers of common and unclustered nodes are approximated. Finally, we conduct MATLAB simulations and demonstrate that the analytical results of our model match well with the simulation results.

II. SYSTEM MODEL

Consider a connected VANET on a multi-lane highway with no on or off ramps. We focus on a single lane with lane changes implicitly captured in the adopted mobility model. We choose a single lane from a multi-lane highway instead of a single-lane highway, in order to be more realistic in a highway scenario. Assume that the highway is in a steady traffic flow condition and the change rate in cluster-membership as a measure of external cluster stability.

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B. Node Mobility

The vehicles move according to the microscopic mobility model proposed in [20]. In this model, a distance headway, \( X \), changes according to a discrete-time finite-state Markov chain. The Markov chain has \( N_{\text{max}} \) states corresponding to \( N_{\text{max}} \) ranges of a distance headway. Let \( X_i(m) \in s_i \) denote the event that the \( i \)-th distance headway is in state \( s_i \) at the \( m \)-th time step\(^1\), where \( s_i \in [0, N_{\text{max}} - 1] \) and \( i, m \geq 0 \). The distance headway transits from one state to another according to a tri-diagonal state-dependent transition matrix, denoted by \( M \). Within a time step, a distance headway in state \( j \) can transit to the next state, the previous state, or remain in the same state with probabilities \( p_{j,j+1} \), \( q_j \), or \( r_j \), \( 0 \leq j \leq N_{\text{max}} - 1 \), respectively, where \( q_0 = p_{N_{\text{max}}-1,0} = 0 \) and \( r_j = 1 - p_j - q_j \).

III. EXTERNAL CLUSTER STABILITY

The cluster-overlap state is governed by the distance between two neighboring CHs. As this distance decreases, the CHs approach each other causing the two clusters to overlap. On the other hand, as the distance between CHs increases, the CHs move apart from each other causing the two clusters to become disjoint. The distance between two neighboring CHs is equal to the sum of the distance headways between the two nodes. Label the \( (N_c + 2) \) nodes with IDs \( 0, 1, \ldots, N_c + 1 \), where the following CH has ID 0 and the leading CH has ID \( N_c + 1 \). For notation simplicity,

\(^1\)The length of the range covered by each state is a constant, denoted by \( L_s \), in meters. The range is chosen such that \( L_s \geq \bar{v} \tau \) where \( \bar{v} \) is the maximum relative speed between vehicles and \( \tau \) is the constant time step size. The mobility model parameters are described in details in [20].
let $X_c = (X_i)_{i=0}^{N_c}$ be the sequence of distance headways between the two CHs as illustrated in Figure 1, where $X_c(m) = (X_i(m))_{i=0}^{N_c}$, and $\{X_c(m) \in \{s_0, s_1, \ldots, s_{N_c}\}\} \equiv \{X_i(m) \in s_i, \forall i \in [0, N_c]\}$. Consider initially overlapping clusters, i.e., $\sum_{i=0}^{N_c} X_i(0) < 2R$. Two neighboring CHs remain overlapping until $\sum_{i=0}^{N_c} X_i(m) \geq 2R$ at some time step $m$. The sequence of $(N_c+1)$ independent and identically distributed (i.i.d.) distance headways is an $(N_c+1)$-dimensional Markov chain, where each headway, $X_i$, is a birth and death Markov chain as described in Subsection II-B. For clarity, the term state refers to a state in the original Markov chain, $X$, the term super state refers to a state in the $(N_c+1)$-dimensional Markov chain, and the term lumped state refers to a set of super states (to be discussed later in this section). Additionally, parentheses ($\left(\right)$) are used for a sequence, while curly brackets ($\{\}$) are used for a set. A super state in the $(N_c+1)$-dimensional Markov chain is a sequence of size $N_c+1$, in which the $i$th element represents the state (in the 1-dimensional Markov chain) that the $i$th distance headway belongs to. That is, a super state, $(s_0, s_1, \ldots, s_{N_c})$, means that distance headway $X_i$ is in state $s_i \in [0, N_{\text{max}} - 1]$. The sum of $(N_c+1)$ distance headways representing the distance between the two CHs can be calculated from the $(N_c+1)$-dimensional Markov chain. The state space size of the $(N_c+1)$-dimensional Markov chain is equal to $N_{\text{max}}^{N_c+1}$, making it subject to the state-space explosion problem when $N_c$ is large. However, since we are interested in the sum of the $(N_c+1)$ distance headways, the state space can be reduced according to the following theorem.

**Theorem 1:** Let $X$ be a discrete-time, birth-death, irreducible Markov chain with $N_{\text{max}}$ finite states, and let set $X = (X_i)_{i=0}^{N_{\text{max}}-1}$ represent a system of $N$ independent copies of chain $X$. The $N$-dimensional Markov chain that represents the system, $X_c$, is lumpable with respect to the state space partition $\Omega = \{\Omega_0, \Omega_1, \ldots, \Omega_N\}$, such that (s.t.) any two super states in subset $\Omega_i$ are permutations of the same set of states $\forall i \in [0, N_{L} - 1]$, where $N_L = \frac{(N_{\text{max}} + N - 1)!}{N_{\text{max}}!(N - 1)!}$ is the state space size of the lumped Markov chain.

The proof of Theorem 1 and following corollaries are given in the Appendix. Since a lumped state, $\Omega_i = \{(s_0, s_1, \ldots, s_{N-1})\}, 0 \leq i \leq N_L - 1$, contains all super states that are permutations of the same set of states, we can write the lumped state as a set of those states $\Omega_i = \{s_0, s_1, \ldots, s_{N-1}\}$. The stationary distribution of the lumped Markov chain is also reducible [21]. The stationary distribution of the lumped Markov chain can be derived from the stationary distribution of the 1-dimensional Markov chain according to the following Corollary.

**Corollary 1:** Consider a system of $N$ independent copies of a finite, discrete-time, birth-death, irreducible Markov chain, $X$, with stationary distribution $(\pi_i)_{i=0}^{N_{\text{max}}-1}$. The stationary distribution of the lumped Markov chain of Theorem 1, representing the system, $X_c = (X_i)_{i=0}^{N_{\text{max}}-1}$, follows a multi-nomial distribution with parameters $(\pi_i)_{i=0}^{N_{\text{max}}-1}$.

**A. Time to the first change of cluster-overlap state**

Consider two overlapping clusters. At any time instant, the overlapping range between two neighbouring clusters is equal to $2R - \sum_{i=0}^{N_c} X_i(m)\forall m \geq 0$. Therefore, according to Theorem 1, the time variation of the overlapping range between the two clusters can be described by a lumped Markov chain with lumped states $\Omega_0, \Omega_1, \ldots, \Omega_{N_{L} - 1}$ which represents the system, $X_c = (X_i)_{i=0}^{N_{\text{max}}-1}$. Furthermore, divide the lumped states into two sets, $\Omega_{\text{OV}}$ and $\Omega_{\text{NOV}}$. A lumped state $\Omega_i = \{s_0, s_1, \ldots, s_{N_c}\}$ belongs to $\Omega_{\text{OV}}$ and to $\Omega_{\text{NOV}}$ if $\sum_{i=0}^{N_c} s_i < 2R_{\text{OV}}$ and $\sum_{i=0}^{N_c} s_i \geq 2R_{\text{NOV}}$, respectively, where $R_{\text{OV}}$ and $R_{\text{NOV}}$ is the integer number of the states that cover distance headways within $R$ in the distance headways’ 1-dimensional Markov chain. Let the system of the distance headways between the two CHs be initially in super state $I_c$, i.e., $X_c(0) \in I_c$, s.t. $I_c \in \Omega_k \in \Omega_{\text{OV}}, 0 \leq k \leq N_L - 1$. Let the time period until the clusters are no longer overlapping be $T_{ov1}(\Omega_k)$, given that the distance headways between them are initially in states $I_c \in \Omega_k$. Then, this time period is equal to the first passage time $T_{ov1}(\Omega_k)$, where $0 \leq k \leq N_L - 1$. Let $T_{ov1}(\Omega_k) = \min \left\{m > 0; X_c(m) \in (k_0, k_1, \ldots, k_{N_c})\right\}$, $\sum_{i=0}^{N_c} k_i \geq 2N_{R_{\text{OV}}}$, and $X_c(0) \in I_c$. Let $M_c$ be the transition probability matrix of the lumped Markov chain describing $X_c$. One way to find the first passage time is to force the lumped states in $\Omega_{\text{NOV}}$ to
become absorbing, i.e., set the probability of returning to the same lump state, $\Omega_i$, within one time step to zero, $\forall \Omega_i \in \Omega_{NOV}$. Furthermore, let all the lumped states in $\Omega_{NOV}$ be merged into a single absorbing state and let it be the last $(N_L - 1)$th state, where $N_L$ is the number of states in the new absorbing lumped Markov chain. The transition probability matrix of the new absorbing lumped Markov chain, $M_N$, is derived from $M_N$, as follows: $M_N(\Omega_i, \Omega_j) = M_N(\Omega_i, \Omega_k) \forall i, j$, s.t. $\Omega_i, \Omega_j \in \Omega_{OV}$, $M_N(\Omega_i, \Omega_{NL-1}) = \sum_j M_N(\Omega_i, \Omega_j) \forall i, j$, s.t. $\Omega_i \in \Omega_{OV}$ and $\Omega_j \in \Omega_{NOV}$. Let $T_{ov}^i(\Omega_k)$ denote the time interval from the instant that the clusters are initially formed till the first time instant that the cluster-overlap state changes, given that the distance headways are in super state $1_c \in \Omega_k$. The cdf of $T_{ov}^i(\Omega_k)$ is given by
\[
F_{T_{ov}^i}(\Omega_k)(m) = M_N(\Omega_k, \Omega_{NL-1}) + \sum_{\Omega_j \in \Omega_{OV}} M_N(\Omega_k, \Omega_j) F_{T_{ov}^i}(\Omega_j)(m-1),
\]
where $F_{T_{ov}^i}(\Omega_k)(0) = 0$. Equation (1) calculates the cdf of $T_{ov}^i(\Omega_k)$ recursively. Since $F_{T_{ov}^i}(\Omega_k)(m) = \sum_{m=1}^{\infty} F_{T_{ov}^i}(\Omega_k)(m)$, the first term in (1) corresponds to the absorption probability within one time step given that the system is initially in lumped state $\Omega_k$, i.e., $F_{T_{ov}^i}(\Omega_k)(1) = M_N(\Omega_k, \Omega_{NL-1})$. The second term in (1) corresponds to $\sum_{m=2}^{\infty} F_{T_{ov}^i}(\Omega_k)(m)$ which is the absorption probability within $(m-1)$ time steps given that the system transited from $\Omega_k$ to $\Omega_j \in \Omega_{OV}$ within one time step.

The size of the state space of the lumped Markov chain can still be large with an increased number of nodes between the two CHs, since $N_L = (N_{max} + N_c)! / [N_c + 1]! [N_{max} - 1]!$. However, the state space of the absorbing lumped Markov chain, needed to compute the time period until the overlap state changes between the two neighboring CHs, is bounded according to the following Corollary.

**Corollary 2:** Consider a system of $N$ independent copies of an irreducible Markov chain according to Theorem 1, and let the event of interest be that the sum of the states of the $N$ chains be larger than a deterministic threshold $N_{th}$. The absorbing lumped Markov chain, required to obtain the first occurrence time of the event of interest, has a state space that is bounded by a deterministic function of $N_{th}$, when $N > N_{th}$.

Consider the scalability of analyzing a system of $N$ distance headways, $X_N$, to an increased number of distance headways, $N$. Using the lumped Markov chain, the scalability of analyzing system $X_N$ is improved for: i) the steady-state analysis - The problem of finding the stationary distribution of a system of distance headway is of constant computational complexity with respect to $N$ (according to Corollary 1); and ii) the transient analysis (i.e., the first passage time analysis) - The computational complexity of the first passage time analysis is dependent on the state space size of the considered Markov chain. According to Corollary 2, the state space size of the absorbing lumped Markov chain is upper bounded by the total number of integer partitions of all integer that are less than $N_{th}$ as discussed in Appendix A.3. Figure 3 shows the state space reduction using the proposed lumped Markov chain.

In this subsection, we focus on the time interval from the instant that two partially overlapping neighboring clusters are formed till the time instant that they no longer overlap. Given an initial super state of the two neighboring clusters at the end of the cluster formation stage, consider the following: i) a proactive re-clustering procedure in which re-clustering is triggered after a fixed period of time, say $\Delta t$ seconds from the cluster formation; and ii) a reactive re-clustering procedure in which re-clustering is triggered when the cluster-overlap state changes. In i), the probability that the overlap state changes between the two overlapping neighboring clusters before re-clustering is triggered is equal to $F_{T_{ov}^i}(\Omega_k)(\Delta t)$. In ii), the re-clustering period is equal to $T_{ov}^i(\Omega_k)$ with the cdf calculated by (1). Up until now, we have considered a pair of neighboring clusters in a specific super state when they are initially formed. In reality, the initial state of a pair of neighboring clusters is a random variable. For a given $N_c$, since the distance headways are stationary when the clusters are formed, the probability that two overlapping neighboring clusters are initially in lumped state $\Omega_i$ is given by $\bar{\Omega}_i \left[ \sum_{\Omega_j \in \Omega_{OV}} \bar{\Omega}_j \right]$ where $\bar{\Omega}_i$ is given by (22) in Appendix A.2. Using the law of total probability, the cdf of the time for the first change in overlap state to occur between two initially overlapping clusters is given by
\[
F_{T_{ov}^i}(m) = \frac{\sum_{\Omega_i \in \Omega_{OV}} \bar{\Omega}_i F_{T_{ov}^i}(\Omega_i)(m)}{\sum_{\Omega_i \in \Omega_{OV}} \bar{\Omega}_i}, \quad m = 1, 2, \ldots
\]
B. Time period between successive changes of cluster-overlap state

In the preceding subsection, we have analysed the time interval during which two neighboring clusters remain overlapping since the clusters are formed. During this time interval, the cluster-overlap state remains unchanged. Suppose two neighboring clusters overlap in cluster formation and the overlap state changes at time $T_{ov1}(<\Delta t)$ and becomes non-overlapping. The cluster-overlap state may change again before re-clustering is triggered. As a result, the time period between two consecutive changes of cluster-overlap state equals i) the cluster-overlapping time period when the overlap state changes from overlapping to non-overlapping, plus ii) the cluster-non-overlapping time period when the overlap state changes from non-overlapping to overlapping. During a cluster-overlapping or cluster non-overlapping time periods, the cluster-overlap state remains unchanged indicating how long the cluster remains externally stable.

1) Cluster-overlapping time period:
The second cluster-overlapping time period may not be equal to $T_{ov1}$ since the initial state may not be the same as that when the clusters are initially formed. We refer to this period as cluster overlapping period, denoted by $T_{ov}$.

To derive the distribution of $T_{ov}$, the same approach used to find the distribution of $T_{ov1}$ can be used. Notice that the absorbing lumped Markov chain is the same as that used to calculate the distribution of $T_{ov1}$. The only difference is the distribution of the initial state, $I_c$. One way to find the distribution of $I_c$ at the time when the second overlapping state occurs is as follows:

- Make the lumped states in set $\Omega_{NOV}$ absorbing, without combining them into one absorbing state. The corresponding transition probability matrix, $M''\omega_i$, is equal to $M\omega_i$, with $M''\omega_i(\Omega_j,\Omega_k) = 0$ and $M''\omega_i(\Omega_j,\Omega_j) = 1 \forall i, j, s.t. \Omega_j \in \Omega_{NOV}$.
- Form another absorbing Markov chain by making the lumped states in set $\Omega_{OV}$ absorbing, without combining them into one absorbing state. The corresponding transition probability matrix, $M''\omega_i$, is equal to $M\omega_i$, with $M''\omega_i(\Omega_j,\Omega_k) = 0$ and $M''\omega_i(\Omega_j,\Omega_j) = 1 \forall i, j, s.t. \Omega_j \in \Omega_{OV}$.
- Calculate the absorbing probability $\phi_i$ for each absorbing lumped state $\Omega_i \in \Omega_{OV}$ by

$$\phi_i = \sum_{\Omega_j \in \Omega_{NOV}} \psi_j \lim_{m \to \infty} M''\omega_i(\Omega_j,\Omega_i).$$

The probability that the distance headways between the two neighboring clusters are in state $\Omega_i \in \Omega_{OV}$ at the time when the second overlapping state occurs is equal to $\phi_i$. Therefore, the cdf of the cluster-overlapping period is given by

$$F_{T_{ov}}(m) = \sum_{\Omega_i \in \Omega_{OV}} \phi_i F_{T_{ov1}}(\Omega_i)(m), m = 1, 2, \ldots$$

where $F_{T_{ov1}}(\Omega_i)(m)$ is given by (1). However, using this approach, we lose the advantage of having a single absorbing state and, therefore, a bounded state space (according to Corollary 2). We propose to approximate the distribution of the system initial state at the time when the second overlapping state occurs, $\phi_i$, as follows

$$\phi_i \approx \frac{\tilde{U}_i M_N(\Omega_i, \Omega_{N\omega_i} - 1)}{\sum_{\Omega_i \in \Omega_{NOV}} \tilde{U}_i M_N(\Omega_i, \Omega_{N\omega_i} - 1)}.$$  

The approximated $\phi_i$ for lumped state $\Omega_i(\in \Omega_{OV})$ is equal to its stationary probability weighted with the absorption probability within one time step. Notice that this weight eliminates all the lumped states $\Omega_i \in \Omega_{OV}$ that are not directly accessible from states in $\Omega_{NOV}$. Figure 2 illustrates an example for a lumped Markov chain, where the directly accessible lumped states are those connected by solid lines, i.e. $\Omega_{OV1}$ and $\Omega_{NOV1}$. When the overlapping state of two neighboring clusters changes from non-overlapping to overlapping, the only possible states to be reached first are those in $\Omega_{OV1}$.

2) Cluster-non-overlapping time period:
Consider two initially overlapping clusters, the cluster state can change to become non-overlapping and again to become overlapping. The time period between two consecutive changes of cluster-overlap state equals the cluster-non-overlapping time period when the state changes from non-overlapping to overlapping. Neighboring CHs may move apart from each other and the clusters become disjoint. This may result in disruption to intercluster and/or intracluster communications and/or seizure of the cluster membership status from edge CMs. Producing unclustered nodes that may create their own cluster which can trigger re-clustering and increase the clustering cost. Let $T_{nov}$ denote the cluster non-overlapping time period. The same procedure used to calculate the cdf of $T_{ov}$ can be used to derive the cdf of $T_{nov}$, which is given by

$$F_{T_{nov}}(m) = \sum_{\Omega_j \in \Omega_{NOV}} \psi_j F_{T_{nov1}}(\Omega_j)(m), m = 1, 2, \ldots$$

where $F_{T_{nov1}}(\Omega_j)(m)$ is $\tilde{M}_N(\Omega_j, \Omega_{N\omega_i} - 1) + \sum_{\Omega_k \in \Omega_{NOV}} \tilde{M}_N(\Omega_j, \Omega_k) F_{T_{nov1}}(\Omega_k)(m - 1), m \geq 1, \psi_j = \begin{cases} \sum_{\Omega_i \in \Omega_{NOV}} \tilde{U}_i M_N(\Omega_i, \Omega_{N\omega_i} - 1) & \Omega_i \in \Omega_{NOV} \\ \tilde{M}_N(\Omega_j, \Omega_j) & \Omega_j \in \Omega_{NOV} \end{cases}$

and $\tilde{M}_N$ is the probability transition matrix that corresponds to the lumped Markov chain with all states in $\Omega_{OV}$ combined into one absorbing state. That is, $\tilde{M}_N$ is derived from $M_N$ as follows: $\tilde{M}_N(\Omega_j, \Omega_j) = M_N(\Omega_j, \Omega_j) \forall i, j, s.t. \Omega_j, \Omega_i \in \Omega_{NOV}$, $\tilde{M}_N(\Omega_j, \Omega_{N\omega_i} - 1) = \sum_{\Omega_k \in \Omega_{NOV}} M_N(\Omega_j, \Omega_k) \forall j, s.t. \Omega_j, \Omega_i \in \Omega_{NOV}$ and $\tilde{M}_N = \Omega_{NOV}$.

The average cluster-non-overlapping time period is given by [22]

$$E[T_{nov}] = \Psi(I - \tilde{M}_N)^{-1} M_1$$

where $\Psi$ is a row vector of size $N'_{L}$ in which the $j^{th}$ element equals $\psi_j$, $I$ is the identity matrix of size equal to that of $N'_{L}$, and $M_1$ is a column vector of ones with size $N'_{L}$. The second moment of the cluster-non-overlapping time period is given by

$$E[T_{nov}^2] = \Psi(I - \tilde{M}_N)^{-1} \tilde{M}_N^2 (I - \tilde{M}_N)^{-1} M_1.$$  

\[The first and the second moments of the cluster-overlapping period can be calculated similarly by adjusting (8) to (9) to correspond to the absorbing lumped Markov chain with transition matrix $\tilde{M}_N$.
Figure 4. Illustration of the events that cause changes in cluster-membership.

\[ E[T_{nov}^2] = 2\Psi \hat{M}'_{N_\text{CM}} \left( I - \hat{M}'_{N_\text{CM}} \right)^{-2} M_1 + E[T_{nov}]. \]  

(9)

IV. INTERNAL CLUSTER STABILITY

Due to relative vehicle mobility, two events result in changes to the cluster-membership: \( i \) a vehicle leaving the cluster, and \( ii \) a vehicle entering the cluster. Let \( e_{o_i} \) and \( e_{e_i} \) denote the events that a vehicle leaves the cluster from the right side and the left side of the CH, respectively. Let \( e_{r_i} \) and \( e_{l_i} \) denote the events that a vehicle enters the cluster from the right side and the left side of the CH, respectively. Figure 4 illustrates these events. Consider the time for the first change in cluster-membership to occur after cluster formation, and denote this time by \( T_{CM1} \). This time is equivalent to the first occurrence times of one of the four events, i.e., \( T_{CM1} = T(e_{o_i} \cup e_{r_i} \cup e_{e_i} \cup e_{l_i}) \), where \( T(e) \) denotes the first occurrence time of event \( e \). Furthermore, let \( T_{CM1} = T(e_{o_i} \cup e_{r_i}) \) and \( T_{CM1} = T(e_{e_i} \cup e_{l_i}) \) be the first occurrence time of the first change in cluster-membership (after cluster formation) due to a vehicle leaving and entering the cluster from the right and the left side of the CH, respectively. Therefore, \( T_{CM1} = \min \{ T_{CM1,1}, T_{CM1,2} \} \). Since \( T_{CM1,1} \) and \( T_{CM1,2} \) are independent, the cdf of the time for the first change in cluster-membership to occur after cluster formation is given by 

\[ F_{T_{CM1}}(m) = 1 - (1 - F_{T_{CM1,1}}(m))(1 - F_{T_{CM1,2}}(m)). \]

Notice that \( T_{CM1,1} \) and \( T_{CM1,2} \) are i.i.d. Therefore, we focus on calculating only one of them, say \( T_{CM1,1} \).

A. Time to the first change of cluster-membership

Let \( N_{CM} \) be the number of CMs on the right side of the CH, and assume that \( N_{CM} > 0 \). Let \( X_{CM} = \{ X_i \}_{i=0}^{N_{CM}} \) be the set of nodes of the CH and the CM nodes as illustrated in Figure 5, where \( X_{CM} \subseteq \{ s_0, s_1, \ldots, s_{N_{CM}} \} \) and \( s_i \forall i \in [0, N_{CM}] \). The system, \( X_{CM} \), can be represented by a \((N_{CM} + 1)\)-dimensional Markov chain. Suppose that set \( X_{CM} \) is in super state \( I_{CM} = (k_0, k_1, \ldots, k_{N_{CM}}) \) when the clusters are initially formed, s.t., \( \sum_{i=0}^{N_{CM}-1} k_i < N_R \), \( \sum_{i=0}^{N_{CM}} k_i \geq N_L \), and \( I_{CM} \in \Omega_k \). Let the time period until a node enters/leaves the cluster from one side be \( T_{CM1,1}(\Omega_k) \), given that \( X_{CM} \in I_{CM} \in \Omega_k \). Then this time period is equal to the first passage time for the system, \( X_{CM} \), to transit from state \( I_{CM} \) to a super state \((k'_0, k'_1, \ldots, k'_{N_{CM}})\) such that \( \sum_{i=0}^{N_{CM}} k'_i < N_R \).

![Figure 5. A cluster with \( N_{CM} = 3 \) and \( X_{CM} = \{ X_0, X_1, X_2, X_3 \} \).](image)

(i.e., a node enters the cluster) or \( \sum_{i=0}^{N_{CM}-1} k'_i \geq N_R \) (i.e., a node leaves the cluster). That is, \( T_{CM1,1}(\Omega_k) = \min \{ m > 0; X_{CM}(m) \in (k'_0, k'_1, \ldots, k'_{N_{CM}}), \{ \sum_{i=0}^{N_{CM}} k'_i < N_R \cup \sum_{i=0}^{N_{CM}-1} s_i \geq N_R \} \mid \mathbb{X}_{CM} \in I_{CM} \} \).

Since the change in cluster-membership occurs at the edge of the cluster, the value of \( X_{CM} \) in the system, \( X_{CM} \), is critical to identify the change. Notice that initially, the distance headway \( X_{CM} \) can only be in a state \( k_{N_{CM}} \in [N_R - \sum_{i=0}^{N_{CM}-1} k_i, N_{max}] \). Therefore, we propose to lump the \((N_{CM} + 1)\)-dimensional Markov chain into partitions (lumped states) \( \Omega_1, \Omega_2, \ldots, \Omega_{N_L-1} \), such that each lumped state \( \Omega_k = \{ (s_0, s_1, \ldots, s_{N_{CM}}) \} \) contains all super states that have the first \( N_{CM} \) states, i.e., \((s_0, s_1, \ldots, s_{N_{CM}})\), as permutations of other \(^4\). Let us refer to this chain as \textit{edge lumped Markov chain}. Furthermore, divide the lumped states into three sets, \( \Omega_1, \Omega_2 \), and \( \Omega_3 \), such that a lumped state \( \Omega_k = \{ (s_0, s_1, \ldots, s_{N_{CM}}) \} \) belongs to i) \( \Omega_1 \), if \( \sum_{i=0}^{N_{CM}-1} s_i < N_R \); and ii) \( \Omega_2 \), if \( \sum_{i=0}^{N_{CM}-1} s_i \geq N_R \), such that \( \Omega_k \in \Omega_1 \) and \( \Omega_k \in \Omega_2 \). For \( \Omega_3 \), if \( \sum_{i=0}^{N_{CM}-1} s_i < N_R \), \( \sum_{i=0}^{N_{CM}-1} s_i \geq N_R \), and \( \sum_{i=0}^{N_{CM}-1} s_i < N_R \). Let \( M_{N_{CM}} \) be the transition probability matrix of the described lumped Markov chain. The time for the first cluster-membership change to occur, \( T_{CM1,1}(\Omega_k) \), is the first passage time for system \( X_{CM} \) to transit from super state \( I_{CM} \in \Omega_k \in \Omega_1 \) to any state in \( \Omega_L \) (i.e., when a node leaves the cluster) or \( \Omega_3 \) (i.e., when a node enters the cluster). To find the distribution of \( T_{CM1,1}(\Omega_k) \), we force the lumped states in \( \Omega_2 \) and \( \Omega_3 \) to become one absorbing state. Following the same steps as in Section III, the cdf of \( T_{CM1,1}(\Omega_k) \) can be derived

\[ F_{T_{CM1,1}(\Omega_k)}(m) = \tilde{M}_{N_{CM}}(\Omega_k, \Omega_{N_L-1}) \]

\[ + \sum_{j \in \Omega_1} \tilde{M}_{N_{CM}}(\Omega_k, \Omega_j) F_{T_{CM1,1}(\Omega_j)}(m - 1), \quad m \geq 1 \]  

(10)

where \( \tilde{M}_{N_{CM}} \) is the probability transition matrix of the new absorbing lumped Markov chain with \( N_L \) states, such that the \( (N_L - 1) \)th state is the single absorbing state containing all states in \( \Omega_1 \) and \( \Omega_2 \).

For a random initial state of \( X_{CM} \), the probability that \( X_{CM} \) is initially in lumped state \( \Omega_1 = \{ (s_0, s_1, \ldots, s_{N_{CM}}) \} \) is given by

\[ \frac{\sum_{j \in \Omega_1} \pi_{i=0}^{N_{CM}} K_i}{\sum_{i=0}^{N_{CM}-1} \pi_{i=0}^{N_{CM}} k_i} \]

\[ K_i = N_R - \sum_{u=0}^{N_{CM}-1} s_u, \]

\[ i \]

\[ \text{When } N_{CM} = 0, \text{ the problem reduces to a single distance headway, with only the event of a node entering the cluster causing the cluster-membership change. In this case, the first passage time analysis for one dimensional chain can be used.} \]

\[ ^4\text{Since the } (N_{CM} + 1) \text{-dimensional Markov chain is lumpable into partitions } \Omega_1, \Omega_2, \ldots, \Omega_{N_L-1}, \text{ each lumped state } \Omega_k = \{ (s_0, s_1, \ldots, s_{N_{CM}}) \} \text{ contains all super states that are permutations of each other according to Theorem 1. Then, it is lumpable into partitions that are subsets of } \Omega_0, \Omega_1, \ldots, \Omega_{N_L-1} \text{.} \]
where $\Omega_i$ is the stationary distribution of lumped state $\Omega_i = \{(s_0, s_2, \ldots, s_{N_{CM} - 1})\}$ of the $N_{CM}$-dimensional Markov chain lumped according to Theorem 1. Hence, the cdf of the time interval between the time instant when the cluster is initially formed till the first cluster-membership change is given by

$$F_{T_{CM}}(m) = \frac{1}{\sum_{i} \delta_i} \sum_{i,j \in \Omega_i} \pi_{i,j} \sum_{k=K_i}^{N_{CM}} \pi_k \lim_{m \to \infty} M^{(m)}_{N_{CM}}(\Omega_i, \Omega_j),$$

where $(i, s_{N_{CM}})$ is the state index of the distance headway of the $N_{CM}^{th}$ CM in the $i^{th}$ lumped state.

### B. Time period between successive changes of cluster-membership

In the previous subsection, we have analysed the time interval from initial cluster formation to the first cluster-membership change. In order to have a better measure of internal cluster stability, we analyse the time interval between two successive cluster-membership changes in this subsection. Let $T_{CM}$ denote the time interval between two consecutive membership changes of a cluster. Notice that the cluster-membership change rate, i.e., the rate at which nodes enter or leave the cluster, is the reciprocal of $T_{CM}$. We focus on one side of the cluster in this subsection, since a similar derivation for the other side can be done.

To derive the distribution of $T_{CM}$, the first step is to find the distribution of $I_{CM}$ at the time when the first cluster-membership change occurs. In order to do this, first we make the lumped states in sets $\Omega_E$ and $\Omega_I$ of the lumped Markov chain absorbing, without combining them into one state. The result is an absorbing markov chain and let $M_{CM}$ be its probability transition matrix. Then the probability of absorption in lumped state $\Omega_i \in \Omega_E$ and the probability of absorption in lumped state $\Omega_i \in \Omega_L$ are given respectively by

$$\psi_{e,i} = \frac{1}{\sum_{j} \delta_j} \sum_{j \in \Omega, j' \in \Omega_i} \pi_{i,j} \sum_{k=K_i}^{N_{CM}} \pi_k \lim_{m \to \infty} M^{(m)}_{N_{CM}}(\Omega_i, \Omega_j)$$

and

$$\psi_{l,i} = \frac{1}{\sum_{j} \delta_j} \sum_{j \in \Omega, j' \in \Omega_i} \pi_{i,j} \sum_{k=K_i}^{N_{CM}} \pi_k \lim_{m \to \infty} M^{(m)}_{N_{CM}}(\Omega_i, \Omega_j)$$

where $M^{(m)}_{N_{CM}}(\Omega_i, \Omega_j)$ denotes the $(\Omega_i, \Omega_j)^{th}$ entry of the $m^{th}$ power of matrix $M_{N_{CM}}$. Note that $\sum_{\Omega_i, \Omega_j \in \Omega_E} \psi_{e,i}$ and $\sum_{\Omega_i, \Omega_j \in \Omega_L} \psi_{e,i}$ are the probabilities that the first cluster-membership change occurs due to a vehicle entering the cluster and leaving the cluster, respectively. When calculating the time interval between successive cluster-membership changes, the examined system changes. Let $\mathcal{X}_{CM_E}$ and $\mathcal{X}_{CM_L}$ be the systems of distance headways of the CH and the nodes on one side of the cluster when the first cluster-membership change occurs due to a node entering the cluster and a node leaving the cluster, respectively. For example, if system $\mathcal{X}_{CM}$ is absorbed in lumped state $\Omega_i = \{(s_0, s_1, \ldots, s_{N_{CM}})\}$, then the initial lumped state for system $\mathcal{X}_{CM_E}$ is $\{(s_0, s_1, \ldots, s_{N_{CM} - 1})\}$ if $\Omega_i \in \Omega_L$ and the initial lumped state for system $\mathcal{X}_{CM_L}$ is $\{(s_0, s_1, \ldots, s_{N_{CM} + 1})\}$ if $\Omega_i \in \Omega_E$, where $s_{N_{CM} + 1} = \text{max}_1$.

### V. NUMBERS OF COMMON CMS AND UNCLUSTERED NODES BETWEEN CLUSTERS

In Section III, the time for the first change in cluster-overlap state along with the cluster-overlapping and cluster-non-overlapping time periods are studied. Despite the importance of the change in overlap-state as a measure of external cluster stability, it is a binary metric. A quantitative metric that describes in detail the level of external stability is desired. One quantitative measure is the number of nodes located between the clusters. That is, the number of nodes shared between overlapping clusters and the number of nodes left unclustered between disjoint clusters. The number of common nodes between neighboring clusters is an indicator of the level of intercluster communication interference that can occur during the overlapping period. On the other hand, during the non-overlapping period, the number of unclustered nodes between disjoint clusters is an indicator of the portion of network nodes that are left unserved by the clustered structure.

Given initially overlapping neighboring clusters, vehicles can enter and leave the overlapping/unclustered region. Additionally, the cluster-overlap state may change over time. Therefore, in this section we investigate the system of two neighboring clusters in terms of the change of the numbers of common CMs and unclustered nodes between the two clusters along with the change in the cluster-overlap state. Since the system of distance headways between the neighboring clusters, $\mathcal{X}_{CM}$, constructs a finite irreducible lumped Markov chain, there exists an infinite sequence of cluster-overlapping and cluster-non-overlapping time periods [22]. Therefore, the overlap state between clusters fluctuates between overlapping and non-overlapping scenarios.
the clusters are initially overlapping, then \( \eta(0) = -1 \) and \( \zeta_k = T_{k,0}^{\text{ov}} \), and \( \theta_k = T_{k,0}^{\text{ov}} \), i.e., \( k \)th cluster-overlapping period and the \( k \)th cluster-non-overlapping period, respectively. We assume that the \( T_{k,0}^{\text{ov}} \)'s are i.i.d. with cdf (5) and the \( T_{k,0}^{\text{ov}} \) time periods are i.i.d. with cdf (7) and they are independent of one another\(^5\). The \( k \)th cycle is composed of \( \zeta_k \) and \( \theta_k \).

A. Node interarrival time during an overlapping/non-overlapping period

During an overlapping/non-overlapping period, vehicles enter and leave the overlapping/unclustered region resulting in a change in the number of common/unclustered nodes between neighboring clusters. Consider two overlapping clusters. A vehicle can enter the overlapping region from either of the clusters. Let \( T_i \) and \( T_{i1} \) be the first arrival time and the interarrival time of nodes to the overlapping region, respectively. We are interested in the arrival times that cause an increase in the number of common nodes in the two clusters. The time for the first node entering the overlapping region is \( T_i = \min(T(e_{i,1}), T(e_{i,2})) \), where \( e_{i,1} \) is the event that a vehicle enters the following cluster from the right side of its CH, and \( e_{i,2} \) is the event that a vehicle enters the leading cluster from the left side of its CH as illustrated in Figure 7(a). Note that \( T(e_{i,1}) \) and \( T(e_{i,2}) \) have the same probabilistic behaviors. We consider that \( T(e_{i,1}) \) and \( T(e_{i,2}) \) are independent when the number of common nodes between clusters is a small fraction of the total number of nodes in the two clusters.

The times, \( T(e_{i,1}) \) and \( T(e_{i,2}) \), can be calculated independently by applying the first passage time analysis on two edge lumped Markov chains, each identifying the hop edge node of its corresponding cluster, as in Subsection IV-A. However, we propose to approximate the distributions of \( T(e_{i,1}) \) and \( T(e_{i,2}) \) by calculating them from a fully lumped Markov chain with the initial distribution calculated from the state space of the edge lumped Markov chain. Since the distributions of \( T(e_{i,1}) \) and \( T(e_{i,2}) \) are the same, we will focus on one of them only, say \( T(e_{i,1}) \). Let \( S_E \) be a set of states of the edge lumped Markov chain for a cluster with \( N_{\text{CM}} \) nodes, such that a lumped state \( \Omega = \{s_0, s_1, \ldots, s_{N_{\text{CM}}} \} \) belongs to \( S_E \) if \( \sum_{i=0}^{N_{\text{CM}}-1} s_i < N_R \) and \( \sum_{i=0}^{N_{\text{CM}}-1} s_i \geq N_R \). Let \( \{\pi_{E,i}\}_{i=1}^{\frac{1}{2}E} \) be the stationary distribution of the edge lumped Markov chain. Furthermore, divide the lumped states of the fully lumped Markov chain representing system \( X_{\text{CM}} \) into two sets, \( \Omega_R \) and \( \Omega_{R'} \). A lumped state \( \Omega = \{s_0, s_1, \ldots, s_{N_{\text{CM}}} \} \) belongs to \( \Omega_R \) if \( \sum_{i=0}^{N_{\text{CM}}-1} s_i < N_R \) and to \( \Omega_{R'} \) otherwise. Let \( T(e_{i,1}, \Omega_R) \) be the first occurrence time of event \( e_{i,1} \) given that system \( X_{\text{CM}} \) is initially in lumped state \( \Omega = \{s_0, s_1, \ldots, s_{N_{\text{CM}}} \} \) belongs to \( \Omega_R \).

In order to calculate the probability distribution of node interarrival time to the overlapping region, the probability distribution of the state of the system when a node first enters the cluster needs to be calculated. Consider a cluster with \( N_{\text{CM}} - 1 \) nodes at time zero. When a node enters the cluster, system \( X_{\text{CM}} \) representing the \( N_{\text{CM}} \) CMs can only be in an edge lumped state \( \Omega = \{s_0, s_1, \ldots, s_{N_{\text{CM}}} \} \) s.t. the first \( N_{\text{CM}} \) states construct a lumped state, \( \Omega_k = \{s_0, s_1, \ldots, s_{N_{\text{CM}}-1} \} \), in a fully lumped Markov chain for system \( X_{\text{CM}} \), that satisfies i) \( \Omega_k \in \Omega_R \) and ii) \( M_{N_{\text{CM}}}^{\text{CM}} (\Omega_k, \Omega_{k-1}) > 0 \). That is, \( \Omega_k \in \Omega_R \) is directly accessible from a lumped state in \( \Omega_{R'} \). As a result, the node interarrival time to the overlapping region from one cluster can be approximated by

\[
F_{T(e_{i,1})}(m) \approx \sum_{\Omega_j \in \Omega_R} \omega_j F_{T(e_{i,1}, \Omega_j)}(m), \quad m \geq 1
\]

where \( \omega_j = \sum_{\Omega_j \in \Omega_R} \pi_{E,i}^{\text{CM}} \) is the initial probability distribution of states \( \Omega_j \in \Omega_R \) and \( f_{0}(\Omega_i) = \sum_{\Omega_j \in \Omega_i} f_{0}(\Omega_j) \) is a function that maps a lumped state from edge lumped markov chain to the corresponding one in the fully markov chain, note that \( \omega_j = 0 \) if \( S_E \nsubseteq S \).

When two clusters become disjoint, vehicles enter and leave the unclustered region. Let us consider the node interdeparture time from the unclustered region that causes the number of unclustered nodes to decrease, denoted by \( T_{io} \). Nodes can leave the unclustered region and enter either of the two clusters. It can be concluded that the time for a node to leave the unclustered region is equal to the minimum of two time intervals \( T(e_{i,1}) \) and \( T(e_{i,2}) \), as illustrated in Figure 7(b). Notice that the events that cause the node departure from the unclustered region during a non-overlapping period are the same as those causing the node arrival to the overlapping region during the overlapping period. Therefore, the distribution of \( T_{io} \) can be calculated accordingly.

\[ F_{T_{io}}(m) = 1 - (1 - F_{T(e_{i,1})}(m))^2. \]

\(^{5}\)Index \( k \) is dropped from \( T_{k,0}^{\text{ov}} \) and \( T_{k,0}^{\text{ov}} \) to refer to an arbitrary overlapping and non-overlapping period, respectively.
B. Steady-state distributions of the numbers of common CMs and unclustered nodes

In this section, we investigate the limiting behavior of the external cluster stability. Considering clusters initially formed to be partially overlapping, we examine the external cluster stability under the assumption that cluster maintenance is not implemented. That is, we want to answer two questions: After a long time, what is the probability that two neighboring clusters are overlapping (non-overlapping)? What is the probability distribution of the number of common CMs (unclustered nodes) in the overlapping (unclustered) region?

The first question can be answered using the theory of alternating renewal process. The limiting overlapping and non-overlapping probability is given by

\[ P_{ov} = \frac{E[T_{ov}]}{E[T_{ov}]+E[T_{nov}]}, \]

and

\[ P_{nov} = \frac{E[T_{nov}]}{E[T_{ov}]+E[T_{nov}]}, \]

respectively [23]. For the second question, we propose to model the problem as a storage buffer with a two-state random environment [24]. The buffer content represents the number of nodes in the overlapping/unclustered region between neighboring clusters. The two random states of the buffer are the overlapping and the non-overlapping states which fluctuate according to the alternating renewal process as described earlier. Let \( N_i(\zeta_k) \) be the numbers of nodes entering (leaving) the buffer during the \( k \)-th overlapping period (non-overlapping period), respectively. Let \( N_i(\Delta t) \) be the numbers of nodes entering (leaving) the overlapping (unclustered) region during an arbitrary time period, \( \Delta t \), respectively. The numbers \( N_i(\Delta t) \) and \( N_o(\Delta t) \) are point processes corresponding to the i.i.d. interarrival periods \( T_{i1} \) and \( T_{o1} \) and representing the input process (output process) of nodes to (from) the buffer, respectively. The mean and the variance of the input process during an overlapping period and the output process during a non-overlapping period are given by [23]

\[
E[N_i(\zeta_k)] = \frac{E[T_{ov}]}{E[T_{i1}]}, \quad \text{Var}[N_i(\zeta_k)] = \frac{c_{T_{i1}}^2}{E[T_{i1}]}E[T_{ov}], \quad (16)
\]

\[
E[N_o(\theta_k)] = \frac{E[T_{nov}]}{E[T_{o1}]}, \quad \text{Var}[N_o(\theta_k)] = \frac{c_{T_{o1}}^2}{E[T_{o1}]}E[T_{nov}], \quad (17)
\]

respectively, where \( c_{T_{i1}} \) and \( c_{T_{o1}} \) are the coefficients of variation of \( T_{i1} \) and \( T_{o1} \), respectively. Consider the \( k \)-th cycle. The buffer content at the beginning of the cycle is given by \( \delta_k = \left[B_{k-1} + N_i(\zeta_{k-1}) - N_o(\theta_{k-1})\right]^+ \). Assuming that the processes \( N_i(\zeta_{k-1}) \) and \( N_o(\theta_{k-1}) \) are non-decreasing for all \( k \), the buffer content model can be associated with a G/G/1 queue [24]. In the queuing model, the service time of customer \( k-1 \) is \( S_{k-1} = N_i(\zeta_{k-1}) \) and the interarrival time between customers \( k-1 \) and \( k \) is \( A_{k-1} = N_o(\theta_{k-1}) \). Then the buffer content at the beginning of the \( k \)-th cycle is the waiting time of the \( k \)-th customer. Therefore, the buffer content at an arbitrary time step, \( m \), is equal to the virtual waiting time (or the workload) of this G/G/1 queue [24] [25]. The virtual waiting time depicts the remaining service time of all customers in the system at an arbitrary time step. Let \( V(m) \) denote the virtual waiting time (buffer content) at an arbitrary time step \( m \). The relation between the virtual waiting time at the \( n \)-th time step and the customer waiting time at the beginning of a cycle is given by [24]

\[
V(m) = \left[B_n(m) + S_n(m) - m + \sum_{k=1}^{n(m)-1} A_k\right]^+ \quad (18)
\]

where \( n(m) = \max\{k \geq 0 : \sum_{k=1}^{k} A_k \leq m\} \).

To find the limiting probability distribution of the buffer content (i.e., the number of common/unclustered nodes between two neighboring clusters) a diffusion approximation is used. The diffusion approximation is a second order-approximation that uses the first two moments of the service and interarrival times of the G/G/1 queue [26]. Let \( \rho = E[S_i]/E[A_k] \) be the intensity factor. A steady-state distribution of the buffer content exists if \( \rho < 1 \) and it is approximated by a geometric distribution with parameter equal to \( \left(1 - \frac{\lambda^2}{\lambda^2 - 2\mu}\right) \). The approximated pmf is given by [26] [27]

\[
P_V(n) \approx \left(1 - \frac{\lambda^2}{\lambda^2 - 2\mu}\right) \left(\frac{\lambda^2}{\lambda^2 - 2\mu}\right)^n, \quad n \geq 0 \quad (19)
\]

where \( \mu = \rho - 1 \) and \( \lambda^2 = \frac{E[S_i^2]}{E[A_k]} \) which can be calculated from (16) and (17). The limiting probability distribution of the numbers of common CMs and unclustered nodes between the two clusters can be described by the pmf (19) with probability \( P_{ov} \), and \( P_{nov} \), respectively. Let \( P_{C0} \) and \( P_{C1} \) denote the limiting probabilities that there are zero common CMs and zero unclustered nodes between neighboring clusters, respectively. These probabilities are given by \( P_{C0} = P_{ov}P_V(0) + P_{nov} \), and \( P_{U0} = P_{nov}P_V(0) + P_{ov} \).

VI. NUMERICAL RESULTS AND DISCUSSION

This section presents numerical results for the analysis of the proposed external and internal cluster stability metrics. The external cluster stability metrics are the time to the first change of cluster-overlap state, \( T_{ov1} \), and the time interval between successive changes of cluster-overlap state (cluster-overlapping period, \( T_{ov} \), and cluster-non-overlapping period, \( T_{nov} \)). The internal cluster stability metrics are the time to the first change of cluster-membership, \( T_{CM1} \), and the time between successive cluster-membership changes, \( T_{CM} \). Additionally, numerical results are presented for pmfs of the steady-state numbers of common CMs and unclustered nodes between two neighboring clusters. We consider a connected VANET in three traffic flow conditions, uncongested, near-capacity, and congested, each corresponding to a set of parameters listed in Table I. For values of \( N_e \) and \( N_{CM} \) at the 6-th time step, we simulate a simple weighted clustering algorithm, where CHs are chosen with the minimum average relative speed to its one-hop neighbors, such that each vehicle belongs to a cluster and no two CHs are one-hop neighbors (i.e., similar to the use of mobility information for clustering in [10], [11]). The distance headways of vehicles on the highway follow a truncated exponential, gamma, and Gaussian distributions for the uncongested, near-capacity, and congested traffic flow conditions, respectively. The vehicles’ speeds are i.i.d. and are normally distributed with mean 100 kilometer per hour and standard deviation of 10 kilometer per hour [28]. Figure 8 plots the probability distributions of \( N_e \) and \( N_{CM} \) for the resulting clusters from simulating the clustering algorithm. Initially, we

Table I. System parameters in simulation and analysis

<table>
<thead>
<tr>
<th>Traffic flow condition</th>
<th>D(veh/km)</th>
<th>E[N_{CM}]</th>
<th>E[N_e]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncongested</td>
<td>9</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Near-capacity</td>
<td>26</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Congested</td>
<td>42</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>R (meter)</td>
<td>N_{max}</td>
<td>X_{CM}(0)</td>
<td>X_{CM}(0)</td>
</tr>
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<td>160</td>
<td>9</td>
<td>{0,1,1,0,2}</td>
<td>{1,1,1,1,5}</td>
</tr>
</tbody>
</table>
set \( N_c \) to its average value from the cluster formation results. For \( D = 42 \) vehicles per kilometer (veh/km), we set \( I_c \) and \( I_{CM} \) to the states with highest probability of occurrence at the cluster formation stage. The Markov-chain distance headway model has the following parameters: \( N_{\text{max}} = 9 \), each state covers 20 meters range of distance headways, the time step is equal to 2 seconds, and the transition probabilities are tuned according to the results in [20]. Based on these parameters, we generate time series of distance headway data according to the microscopic mobility model, using MATLAB. Each simulation consists of 20,000 iterations.

Figure 9 compares the distribution of the state of system \( X_c \), when the second overlapping state occurs, calculated using the exact derivation (4) and the proposed approximation (6). The values on the x-axis represent arbitrary IDs given to the lumped states \( \Omega_i \in \Omega_{OV} \). The results from the proposed approximation shows close agreement with the exact and the simulation results. Figure 10 plots the pmf of the time interval for the first change in cluster overlapping state, for (a) a given initial state of \( X_c \) and (b) when averaging over random initial states, respectively. The theoretical results for the pmfs of the cluster-overlapping period are calculated from the cdf in (5). The calculated pmf of \( T_{ov} \) in Figure 10(c) is based on the approximation given in Figure 9. The distribution of \( T_{ov1}(\Omega_k) \) changes with \( I_c \) belonging to different lumped states \( \Omega_k \). The distribution of \( T_{ov1} \) describes the average time before the first cluster-overlap change for a randomly picked cluster in the network. When clusters overlap, the cluster-overlapping period is equal to the time period between two successive cluster-overlap state changes (i.e., the time period of invariant cluster-overlap state). Note that the average time for the first change of cluster-overlap state is larger than the average time period between successive changes of cluster-overlap state. When the second overlapping state occurs between neighboring clusters, the clusters state is closer to non-overlapping than that when the clusters are initially formed, on average. That is, the clusters state can only be in the accessible lumped states (\( \Omega_{OV1} \) in Figure 2).

Figure 11 plots the pmf of the time period from the cluster formation till the time step that a first change in cluster-membership occurs for (a) a given initial state \( I_{CM} \) ∈ \( \Omega_k \) for the first change in cluster overlapping state, for (a) a given initial state of \( X_c \) and (b) when averaging over random initial states, respectively. The theoretical results for the pmfs of the cluster-overlapping period are calculated from the cdf in (5). The calculated pmf of \( T_{ov} \) in Figure 10(c) is based on the approximation given in Figure 9. The distribution of \( T_{ov1}(\Omega_k) \) changes with \( I_c \) belonging to different lumped states \( \Omega_k \). The distribution of \( T_{ov1} \) describes the average time before the first cluster-overlap change for a randomly picked cluster in the network. When clusters overlap, the cluster-overlapping period is equal to the time period between two successive cluster-overlap state changes (i.e., the time period of invariant cluster-overlap state). Note that the average time for the first change of cluster-overlap state is larger than the average time period between successive changes of cluster-overlap state. When the second overlapping state occurs between neighboring clusters, the clusters state is closer to non-overlapping than that when the clusters are initially formed, on average. That is, the clusters state can only be in the accessible lumped states (\( \Omega_{OV1} \) in Figure 2).

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Figure 11. The pmfs of (a) the time to the first change in cluster-membership, $T_{CM1}(\Omega_k)$, for $I_{CM} = \{1,1,1,5\} \in \Omega_k$ when the cluster is initially formed; (b) the time to the first change in cluster-membership, $T_{CM1}$; and (c) the time period between two successive cluster-membership changes, $T_{CM}$, when $D = 26$ veh/km.

and (b) a random initial state, and (c) the pmf of the time period between successive cluster-membership changes. The theoretical results for the pmfs of $T_{CM1}(\Omega_k)$ and $T_{CM1}$ are calculated from the cdfs in (10) and (11), respectively. The pmf of the time period between successive cluster-membership changes is calculated from the cdf in (12) and is plotted in Figure 11. The simulation results closely agree with the theoretical calculations. It is observed that, when the first change in cluster membership occurs after the cluster formation, the second change in cluster membership has a higher probability of occurring in a shorter time period. This reflects the effect of a wireless link between a CM and CH fluctuating between connecting and disconnecting states in a short period of time. The impact of this fluctuation can lead to frequent re-clustering that drains the precious VANET radio resources. Some clustering algorithms for VANETs aims to localize the impact of this fluctuation within the clusters [4], [7], [10].

Figure 12 plots the pmf of the first arrival time of nodes into the overlapping region $T_{eov}^{CM}$, for a near-capacity traffic flow condition. The exact theoretical value is calculated from the edge Markov chain as explained in Appendix A.4, whereas the approximated value is calculated from the fully lumped Markov chain using (13). The results show that approximating the node-arrival time to the overlapping/unclustered region using the fully lumped Markov chain is adequate.

Figure 13 plot the pmfs of the cluster-overlapping, $T_{ov}$, time period for different vehicle densities when $N_c$ is set to the average values in Table I. The time interval between successive changes of cluster-overlap state is equal to $T_{ov}$ ($T_{nov}$) when the two clusters are overlapping (disjoint). Notice that the vehicle density has little impact on the distribution of the overlapping/non-overlapping periods when $N_c$ is set to the average value. However, this is not true for all $N_c$.

Figure 14 plots the average cluster-overlapping and the average cluster-non-overlapping time periods for different numbers of nodes between neighbouring clusters, $N_c$. The average values are calculated using (8) and the values of $N_c$ are the clustering results in Figure 8. For a fixed $N_c$, the average cluster-overlapping period is larger for a larger density, whereas the average cluster-non-overlapping period is smaller for a larger density. The reason is that, in a congested traffic flow conditions, the distance headways are small when compared to those in an uncongested traffic flow condition. Therefore, for the same $N_c$, the cumulative distances are smaller for a high density. It should be noted that the large values of average cluster-overlapping time periods for $N_c = 1$ are due to the connected network assumption. Figure 14 shows that, as $N_c$ increases, the average cluster-overlapping period reduces and the average cluster-non-overlapping period increases for the same traffic flow condition.

To investigate the limiting behavior of the number of vehicles in the overlapping/unclustered region, we first calculate the two parameters $\mu$ and $\lambda^2$ for the three vehicle densities. Notice that the distributions (5), (7), and (15) are all conditional on the initial cluster state in terms of $N_c$ and $N_{CM}$. Therefore, in the calculation of $\mu$ and $\lambda^2$, we use the law of total expectation to calculate $E[T_{ov}] = \sum_n P_{N_c}(n)E[T_{ov}(n)]$ and $E[T_{ov}^2] = \sum_n P_{N_c}(n)E[T_{ov}^2(n)]$, where $T_{ov}(n)$ is the cluster-overlapping time period for two clusters separated by $N_c = n$ nodes and $T_{ov}(n)$ is the node interarrival time for a cluster with $N_{CM} = n$ nodes, respectively. The calculations are done for near-capacity and congested traffic flow conditions only. The reason is that the diffusion approximation assumes that the point processes $N_1(\xi_n)$ and $N_2(\theta_n)$ are normally distributed according to the central limit theorem. This assumption is not satisfied for an uncongested traffic flow, due to a relatively small number of vehicles between two clusters as shown in Figure 8. The intensity factor is found to be $\rho = 1.0143$ and $1.3172$ for $D = 26$ and 42 veh/km, respectively. As a result, the steady-state distribution does not exist. However, consider only $N_c \geq E[N_c]$ for both cases, we find that $\rho = 0.33$, and 0.64 for $D = 26$, and 42 veh/km, respectively. Figure 15 plots the steady-state probability distributions for the non-zero number of vehicles in the overlapping/unclustered region when $N_c \geq E[N_c]$ for near-capacity and congested traffic flow conditions. The theoretical results are normalized to the value $1 - P_Y(0)$, since the probability distributions in Figure 15 represent

$^7$Figure 16, in the appendix, plots the pmfs of the time interval between two successive cluster-membership changes for different vehicle densities.

$^8$The pmfs of the cluster-non-overlapping, $T_{nov}$, follow the same trends. So, the plots are omitted due to space limitations.
the non-zero number of common CMs with probability $P_{ov}$ and the non-zero number of unclustered nodes with probability $P_{nov}$. The simulation results closely agree with the theoretical calculations. However, there exist slight differences between simulation and theoretical results especially at the values of $n = 5$ and $n = 8$, for $D = 26$ and 42 veh/km, respectively. This is mainly due to complete overlapping between neighboring clusters. When two clusters completely overlap, i.e., become one hop neighbors, all the nodes between them become common nodes, however, no additional nodes can enter the overlapping region. This is not accounted for in our model. According to many clustering algorithms, when two CHs become one hop neighbors, they merge into a single cluster [7], [8], [10]. Figure 15 shows that the simulation results excluding the complete cluster overlapping data are in closer agreement with the theoretical results in comparison with simulation results that include the complete cluster overlapping data. Additionally, the numerical and simulation results for the limiting probabilities of having zero common CMs and zero unclustered nodes are given in Table II.

The probability distributions of $T_{ov1}, T_{ov}, T_{nov}, T_{CM}$, and $V(m)$ derived in this paper provide indicators for the stability of a cluster in terms of its relation with its CMs and its relation with neighboring clusters. This can be used to enhance network protocol design for VANETs. For example, the derived probability distributions of $T_{ov}$ and $T_{nov}$ can be used to update the transmission codes assigned to different clusters so that the hidden terminal problem caused by cluster overlapping is avoided with a certain desired probability threshold [16]. Additionally, the distribution of the cluster-overlapping period can be utilized to dynamically choose the value of the time threshold used to avoid frequent merging and splitting of neighbouring clusters in VANETs [4], [7], [10]. The time period between successive cluster-membership changes, $T_{CM}$, provides a lower bound on the cluster-membership duration. This is extensively used in the literature for performance evaluation of clustering algorithms [7]–[9]. The probability distribution of $T_{CM}$ can be used to choose the time threshold value that determines when an unclustered node can create its own cluster after it has disconnected from its CH, thus minimizing re-clustering frequency [4], [7], [10]. The limiting probability distribution of the number of common CMs between neighbouring clusters can help determine the amount of additional radio resources (e.g., time slots) that should be allocated to a cluster in order to avoid intercluster interference.

Table II. Limiting probabilities of zero common CMs/unclustered nodes

<table>
<thead>
<tr>
<th>$D$ (veh/km)</th>
<th>Simulation</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{C0}$</td>
<td>26</td>
<td>0.86</td>
</tr>
<tr>
<td>$P_{C0}$</td>
<td>42</td>
<td>0.69</td>
</tr>
<tr>
<td>$P_{U0}$</td>
<td>26</td>
<td>0.57</td>
</tr>
<tr>
<td>$P_{U0}$</td>
<td>42</td>
<td>0.52</td>
</tr>
</tbody>
</table>
VII. CONCLUSION
This paper presents a stochastic analysis of single-hop cluster stability in a highway VANET with focus on a single lane. The time periods of invariant cluster-overlap state and cluster-membership are proposed as measures of external and internal cluster stability, respectively. A stochastic mobility model that describes the time variations of individual distance headways is adopted in the analysis. The system of distance headways that govern the changes in the overlap state and the cluster membership is modeled by a discrete-time lumped Markov chain. The first passage time analysis is employed to derive the distributions of the proposed cluster stability metrics. The analysis provides insights about the time periods during which a cluster is likely to remain unchanged in terms of its cluster-membership and its overlap state with neighboring clusters. Additionally, the limiting probability distributions of the numbers of common and unclustered nodes between neighboring clusters are approximated using queuing theory and diffusion approximation. The probability distributions derived for the proposed cluster stability metrics can be utilized in the development of efficient clustering algorithms for VANETs.

APPENDIX

A.1 Proof of Theorem 1
Let \( M_N = \{M_N(S_i, S_j)\}, 0 \leq S_i, S_j \leq N_{\text{max}}^N - 1 \), be the transition matrix of the \( N \)-dimensional Markov chain that represents the system of \( N \) independent copies of the 1-dimensional Markov chain, \( X \), with transition matrix \( M = \{M(u_i, u_j)\}, 0 \leq u_i, u_j \leq N_{\text{max}} - 1 \). A discrete-time Markov chain with stochastic transition matrix \( M_N \) is lumpable with respect to the partition \( \Omega \) if and only if, for any subsets \( \Omega_i \) and \( \Omega_j \) in the partition, and for any super states \( S_i \) and \( S_2 \) in subset \( \Omega_i \) [21],

\[
\sum_{S \in \Omega_j} M_N(S_1, S) = \sum_{S \in \Omega_j} M_N(S_2, S). \tag{20}
\]

Consider the left hand side (LHS) of (20). Since \( X \) is a birth-death process, the super state \( S_i = (u_0, u_1, \ldots, u_{N-1}) \), \( 0 \leq u_i \leq N_{\text{max}} - 1 \), can transit to any super state in set \( A = \{(u_0', u_1', \ldots, u_{N-1}')\} \), where state \( u_i' \in \{u_i - 1, u_i, u_i + 1\} \), i.e., \( |A| \leq 3^{N_{\text{max}}} \). Let subsets \( A_1 = A \cap \Omega_i \) and \( A_2 = A \cap \Omega_j \). Since \( M_N(S_i, S) = 0 \) \( \forall S \not\in A \), the LHS of (20) reduces to

\[
\sum_{S \in A_1} M_N(S_1, S). \tag{21}
\]

Similarly, for the right hand side (RHS) of (20), the super state \( S_2 = (v_0, v_1, \ldots, v_{N-1}) \), \( 0 \leq v_i \leq N_{\text{max}} - 1 \), can transit to any super state in set \( B = \{(v_0', v_1', \ldots, v_{N-1}')\} \), where state \( v_i' \in \{v_i - 1, v_i, v_i + 1\} \), i.e., \( |B| \leq 3^{N_{\text{max}}} \). Let subsets \( B_1 = B \cap \Omega_i \) and \( B_2 = B \cap \Omega_j \). Since \( M_N(S_2, S) = 0 \) \( \forall S \not\in B \), the RHS of (20) reduces to

\[
\sum_{S \in B_1} M_N(S_2, S). \tag{22}
\]

Consider two sequences, \( S_i \) and \( S_j \), that are permutations of each other, and define \( \varrho(S_i, O_{ij}) = S_j \) to be the permutation operator on sequence \( S_i \) under index order \( O_{ij} \) that gives \( S_j \), i.e., \( S_j = (S_i(O_{ij}(k)))_{k=1}^{N} \). For example, if \( S_i = (1, 0, 2) \) and \( S_j = (0, 2, 1) \), then \( O_{ij} = (2, 3, 1) \).

Let \( S_i' = (u_0', u_1', \ldots, u_{N-1}') \) be a super state in subset \( A_j \). Therefore, \( M_N(S_i, S_i') = \prod_{n=0}^{N-1} M(u_i, u_i') \). Since \( S_i, S_j \in \Omega_i \), there exists an index order \( O_{ij} \), s.t. \( \varrho(S_i, O_{ij}) = S_j \). Additionally, \( \exists S_i'' = (v_0', v_1', \ldots, v_{N-1}') \) s.t. \( \varrho(S_i', O_{ij}) = S_j \).

Note that \( S_j \) and \( S_j'' \) are \( \varrho \)-restricted integers that are integer partitions of an integer that is greater than or equal to \( N_{\text{max}} \). In combinatorics, an integer partition of a positive integer \( n \) is a set of positive integers whose sum equals \( n \). Each member of the set is called a part. An \( N \)-restricted integer partition of an integer \( n \) is an integer partition of \( n \) into exactly \( N \) parts. Therefore, \( \forall S_j' \) \( \in \Omega_j \), \( \{S_i, S_i', \ldots, S_j'\} \) \( \in \Omega_{ij} \).

A.2 Proof of Corollary 1
Consider the tri-diagonal probability transition matrix of the Markov chain, \( X \), as described in Subsection II-B. The stationary distribution of the chain, \( X \), is given by

\[
\pi_i = \prod_{k=0}^{i-1} \left( \frac{p_k}{q_k+1} \right) \pi_0, \quad 1 \leq i \leq N_{\text{max}} - 1 \tag{21}\]

where \( \pi_0 = \left[ 1 + \sum_{i=1}^{N_{\text{max}}-1} \prod_{k=0}^{i-1} \left( \frac{p_k}{q_k+1} \right) \right]^{-1} \). Consider the \( \varrho \)th lumped state \( \Omega_i = \{s_{0}, s_{1}, \ldots, s_{N} \} \). Let \( N_D \) be the number of distinct states in \( \{s_{0}, s_{1}, \ldots, s_{N} \} \) in which \( (u_1, u_2, \ldots, u_{N_D}) \) and \( (n_u_1, n_u_2, \ldots, n_{N_D}) \) are the sequences of distinct states and their corresponding frequencies, respectively, where \( 0 \leq u_i \leq N_{\text{max}} - 1 \) and \( \sum_{i=1}^{N_D} n_u_i = N \). Note that the size of the lumped state is equal to the number of super states that are permutations of each other, i.e., \( 1 \leq |\Omega_i| \leq N_{\text{max}} - 1 \). Therefore, the lumped states result from all possible outcomes of choosing \( N \) states from \( N_{\text{max}} \) different states independently, where choosing state \( s_i \) has the probability \( \pi_i \), \( 0 \leq s_i \leq N_{\text{max}} \). This is a generalization of the Bernoulli trial problem. Hence, the stationary distribution for the lumped state \( \Omega_i \) is given by

\[
\pi_{ij} = \frac{N_{\text{max}}!}{N_D! \prod_{k=1}^{N_D} n_{u_k}!} \pi_{u_k}. \tag{22}\]

That is, the stationary distribution of the lumped Markov chain is multi-nominal, which ends the proof.

A.3 Proof of Corollary 2
Let the lumped state \( \Omega_i = \{s_{0}, s_{1}, \ldots, s_{N} \} \) be a lumped state such that, if the system enters this state, the event of interest occurs. Then, \( \{s_{0}, s_{1}, \ldots, s_{N} \} \) \( \in \Omega_{ij} \). \( \{s_{0}, s_{1}, \ldots, s_{N} \} \) is an integer partition of an integer that is less than or equal to \( N_{\text{th}} \). In combinatorics, an integer partition of a positive integer \( n \) is a set of positive integers whose sum equals \( n \). Each member of the set is called a part. An \( N \)-restricted integer partition of an integer \( n \) is an integer partition of \( n \) into exactly \( N \) parts. Therefore, \( \forall S_j' \) \( \in \Omega_{ij} \), \( \{s_{0}, s_{1}, \ldots, s_{N} \} \) \( \in \Omega_{ij} \).

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Figure 16. The pmf of the time period between successive cluster-

membership changes with vehicle density (a) $D = 9$, (b) $D = 42$ veh/km.

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