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H_∞ Constrained Pareto Suboptimal Strategy for Stochastic LPV Time-Delay Systems

Hiroaki Mukaidani*

Graduate School of Advanced Science and Engineering
Hiroshima University, 1-4-1 Kagamiyama
Higashi-Hiroshima 739-8527, Japan
mukaida@hiroshima-u.ac.jp

Hua Xu

Graduate School of Business Sciences, The University of Tsukuba
3-29-1, Otsuka, Bunkyo-ku, Tokyo 112-0012, Japan
xu@mbaib.gsbs.tsukuba.ac.jp

Weihua Zhuang

Department of Electrical and Computer Engineering
University of Waterloo, 200 University Avenue West
Waterloo, Ontario N2L 3G1, Canada
wzhuang@uwaterloo.ca

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Not only in control problems, but also in dynamic games, several sources of performance degradation, such as model variation, deterministic and stochastic uncertainties and state delays, need to be considered. In this paper, we present an H_∞ constrained Pareto suboptimal strategy for stochastic linear parameter-varying (LPV) time-delay systems involving multiple decision makers. The goal of developing the H_∞ constrained Pareto suboptimal strategy set is to construct a memoryless state feedback strategy set, so that the closed-loop stochastic LPV system is stochastically mean-square stable. In the paper, the existence condition of the extended bounded real lemma is first established via linear matrix inequalities (LMIs). Then, a quadratic cost bound for cost performance is derived. Based on these preliminary results, sufficient conditions for the existence of such a strategy set under the H_∞ constraint are derived by using cross-coupled bilinear matrix inequalities (BMIs). To determine the strategy set, a viscosity iterative scheme based on the LMIs is established to avoid the processing of BMIs. Finally, two numerical examples are presented to demonstrate the reliability and usefulness of the proposed method.

Keywords: Gain-scheduled control; Pareto suboptimal strategy; stochastic linear parameter varying (LPV) system; cross-coupled matrix inequalities (CCMIs); H_∞-constraint.

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*Corresponding author.

1. Introduction

In robust controller design, deterministic and stochastic uncertainties should be addressed, which are caused by the linearization and unmodeled dynamics of the original systems, as well as external uncertainties including stochastic system noise and disturbance. The linear parameter-varying (LPV) system is a reliable mathematical model to capture system variations that are arbitrarily smooth or continuous. It is well known that LPV systems can be accurately represented by using many parameter variations Apkarian *et al.* [1995]; Briat [2015]. Gain scheduling (GS) control techniques are often used to control LPV systems because they can effectively compensate for parameter variations. In particular, an H_∞ GS control problem for stochastic LPV systems is first discussed in [Ku and Wu, 2015] to recover the degradation of stability margin caused by external disturbances. Moreover, the delay-dependent H_∞ control problem of deterministic LPV systems with time-varying state delay has been addressed Zope *et al.* [2012]. The stabilization control problem of the milling process using a state feedback strategy as a practical delay system has been solved. In recent years, with advances in GS control technology, dynamic game problems of stochastic LPV systems involving multiple decision makers have been investigated to guarantee stochastical mean-square stability Mukaidani *et al.* [2018]; Mukaidani and Xu [2018]. However, no time delay in the system model and control is considered in the existing studies. It is well known that delay is a main cause of degraded stability performance for optimized cost values. Therefore, further research is needed to maintain the robust stability in the presence of delay and multiple decision makers.

Consequently, in this paper, we investigate the H_∞ constrained Pareto suboptimal control problem of a stochastic LPV time-delay system involving multiple decision makers. This study is based on the H_2/H_∞ control approach Chen and Zhang [2004] and the Pareto strategy, to ensure the bound weighted sum of linear quadratic costs Engwerda [2005] under stochastically mean-square stable and H_∞ performance. Although the H_∞ constrained Pareto suboptimal strategy for stochastic LPV systems has been studied Mukaidani *et al.* [2018], research on the H_∞ constrained Pareto suboptimal strategy for time-delay systems remains open. There are few results on the H_2/H_∞ control problems for stochastic LPV time-delay systems involving multiple decision makers. The contributions of this paper are as follows: First, an extended existence condition of the existing bounded real lemma for delay LPV systems Ku and Wu [2015] is investigated using the LMI approach. In addition, the existence condition of the quadratic cost bound for each player is established based on the guaranteed cost control technique Moheimani and Petersen [1996]; Rotondo *et al.* [2015]; second, by using these preliminary results, sufficient conditions for the existence of an H_∞ constrained Pareto suboptimal control strategy set are obtained in terms of the cross-coupled bilinear matrix inequalities (BMIs). The H_∞ constrained condition for delay stochastic LPV systems is derived for the first time; third, to solve the cross-coupled BMIs, a viscosity iterative scheme

Xu [2004] based on isolated LMIs is proposed to obtain an LMI solution set corresponding to the strategy set. Furthermore, the strong convergence property of the proposed iterative method can be attained successfully; finally, to demonstrate the effectiveness of the proposed algorithm and the reliability and usefulness of the proposed strategy set, two numerical examples are presented.

Notation: The notations used in this paper are fairly standard: I_n denotes the $n \times n$ identity matrix; $\|\cdot\|$ denotes the Euclidean norm of a matrix; $L^2_F([0,\infty), \mathbb{R}^k)$ denotes the space of nonanticipative stochastic processes $\phi(t) \in \mathbb{R}^k$ with respect to an increasing σ-algebras F_t, $t \geq 0$, satisfying $\mathbb{E}[\int_0^\infty \|\phi(t)\|^2 dt] < \infty$; $C([-h,0]; \mathbb{R}^n)$, $h > 0$, denotes the family of continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \sup_{-h \leq \theta \leq 0} \|\phi(\theta)\|$; $\lambda_{\max}[\cdot]$ and $\lambda_{\min}[\cdot]$ denote its largest and smallest eigenvalue, respectively.

2. Preliminary Results

Consider the following stochastic LPV time-delay system:

$$dx(t) = [A(\theta)x(t) + A_h(\theta)x(t-h) + D(\theta)v(t)]dt + A_p(\theta)x(t)dw(t), \tag{1a}$$

$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{1b}$$

$$z(t) = E(\theta)x(t), \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $v(t) \in \mathbb{R}^{n_v}$ denotes the external disturbance, $z(t) \in \mathbb{R}^{n_z}$ denotes the controlled output, $w(t) \in \mathbb{R}$ denotes a one-dimensional standard Wiener process defined in the filtered probability space, $\theta(t) \in \mathbb{R}^r$ denotes the time-varying parameters, and r denotes the number of time-varying parameters.

In (1), $h \in (0, \infty)$ is the time delay of the stochastic LPV time-delay system, and $\phi(t)$ is a real-valued initial function. It is assumed that, for all $\delta \in [-h, 0]$, there exists scalar $\varepsilon > 0$ such that $\|x(t+\delta)\| \leq \varepsilon \|x(t)\|$ Cao and Lam [2000].

The coefficient matrices in the stochastic LPV time-delay system are parameter-dependent matrices, and can be expressed as

$$[A(\theta) A_h(\theta) A_p(\theta)] = \sum_{k=1}^{M} \alpha_k(t)[A_k \ \ A_{hk} \ \ A_{pk}], \tag{2a}$$

$$D(\theta) = \sum_{k=1}^{M} \alpha_k(t) D_k, \quad E(\theta) = \sum_{k=1}^{M} \alpha_k(t) E_k, \tag{2b}$$

where $\alpha_k(t) \geq 0$, $\sum_{k=1}^{M} \alpha_k(t) = 1$, $M = 2^r$.

As an extension of our previous results in Mukaidani *et al.* [2018], the following theorem can be derived.

Theorem 1. *Consider the stochastic LPV time-delay system in* (1). *Given an attenuation performance level,* $\gamma > 0$, *suppose that there exist matrices* $Z = Z^T > 0$

and $U = U^T > 0$ *satisfying the following LMIs*:

$$\boldsymbol{M}_k^0(Z, U) < 0, \tag{3a}$$

$$\boldsymbol{M}_{k\ell}^0(Z, U) < 0, \tag{3b}$$

where

$$\boldsymbol{M}_k^0(Z, U) := \begin{bmatrix} \Xi_k^0 & ZA_{hk} & ZD_k & A_{pk}^T Z & E_k^T \\ A_{hk}^T Z & -U & 0 & 0 & 0 \\ D_k^T Z & 0 & -\gamma^2 I_{n_v} & 0 & 0 \\ ZA_{pk} & 0 & 0 & -Z & 0 \\ E_k & 0 & 0 & 0 & -I_{n_z} \end{bmatrix},$$

$$\boldsymbol{M}_{k\ell}^0(Z, U) := \begin{bmatrix} \Xi_{k\ell}^0 & ZA_{hk\ell} & ZD_{k\ell} & A_{pk\ell}^T Z & E_{k\ell}^T \\ A_{hk\ell}^T Z & -2U & 0 & 0 & 0 \\ D_{k\ell}^T Z & 0 & -2\gamma^2 I_{n_v} & 0 & 0 \\ ZA_{pk\ell} & 0 & 0 & -2Z & 0 \\ E_{k\ell} & 0 & 0 & 0 & -2I_{n_z} \end{bmatrix},$$

$$\begin{aligned} k &< \ell, \quad k = 1, \ldots, M, \\ \Xi_k^0 &= \Xi_k^0(Z, U) := ZA_k + A_k^T Z + U, \\ \Xi_{k\ell}^0 &= \Xi_{k\ell}^0(Z, U) := ZA_{k\ell} + A_{k\ell}^T Z + U, \\ A_{k\ell} &:= A_k + A_\ell, \quad A_{hk\ell} := A_{hk} + A_{h\ell}, \quad D_{k\ell} := D_k + D_\ell, \\ E_{k\ell} &:= E_k + E_\ell, \quad A_{pk\ell} := A_{pk} + A_{p\ell}. \end{aligned}$$

Then, we have the following results:

(i) *The stochastic LPV time-delay system in* (1) *is stochastically mean-square stable with* $v(t) \equiv 0$;

(ii) *The following inequality holds*:

$$\|z\|_2^2 \le \gamma^2 \|v\|_2^2 + \mathcal{F}(Z, U), \tag{4}$$

where

$$\|z\|_2^2 := \mathbb{E}\left[\int_0^\infty \|z(t)\|^2 dt\right], \quad \|v\|_2^2 := \mathbb{E}\left[\int_0^\infty \|v(t)\|^2 dt\right],$$

$$\mathcal{F}(Z, U) := \mathbb{E}[x^T(0) Z x(0)] + \mathbb{E}\left[\int_{-h}^0 \phi^T(s) U \phi(s) ds\right];$$

(iii) *The worst-case disturbance is given by*

$$v^*(t) = F_\gamma^*(\theta)x(t) = \gamma^{-2}D^T(\theta)Zx(t). \tag{5}$$

Proof. First, define the following Lyapunov–Krasovskii function:

$$V_v(x,t) := x^T(t)Zx(t) + \int_{t-h}^{t} x^T(s)Ux(s)ds, \tag{6}$$

where $Z = Z^T > 0$ and $U = U^T > 0$.

By using Itô formula with infinitesimal generator $\mathscr{L}$, the stochastic differential equation can be obtained as

$$\begin{aligned}
&\mathscr{L}V_v(x,t) + \|z(t)\|^2 - \gamma^2\|v(t)\|^2 \\
&\quad = \xi^T(t)\boldsymbol{M}(Z,U,\theta)\xi(t) - \gamma^2(v(t) - \gamma^{-2}D^T(\theta)Zx(t))^T \\
&\qquad \times (v(t) - \gamma^{-2}D^T(\theta)Zx(t)),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\mathscr{L}V_v(x,t) &= x^T(t)Ux(t) - x^T(t-h)Ux(t-h) \\
&\quad + 2x^T(t)Z[A(\theta)x(t) + A_h(\theta)x(t-h) + D(\theta)v(t)] \\
&\quad + x^T(t)A_p^T(\theta)ZA_p(\theta)x(t), \\
\boldsymbol{M}(Z,U,\theta) &= \begin{bmatrix} \Phi(Z,U,\theta) & ZA_h\theta) \\ A_h^T(\theta)Z & -U \end{bmatrix}, \\
\xi(t) &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}, \\
\Phi(Z,U,\theta) &:= ZA(\theta) + A^T(\theta)Z + U + A_p^T(\theta)ZA_p(\theta) \\
&\quad + \gamma^{-2}ZD(\theta)D^T(\theta)Z + E^T(\theta)E(\theta).
\end{aligned}$$

Hence, if $v(t) = v^*(t)$, we have

$$\mathscr{L}V_v(x,t) + \|z(t)\|^2 - \gamma^2\|v(t)\|^2 \le \xi^T(t)\boldsymbol{M}(Z,U,\theta)\xi(t). \tag{8}$$

On the other hand, using the Schur complement, $\boldsymbol{M}(Z,U,\theta) < 0$ is equivalent to the following inequality:

$$\begin{bmatrix}
ZA(\theta) + A^T(\theta)Z + U & ZA_h(\theta) & ZD(\theta) & A_p^T(\theta)Z & E^T(\theta) \\
A_h^T(\theta)Z & -U & 0 & 0 & 0 \\
D^T(\theta)Z & 0 & -\gamma^2 I_{n_v} & 0 & 0 \\
ZA_p(\theta) & 0 & 0 & -Z & 0 \\
E(\theta) & 0 & 0 & 0 & -I_{n_z}
\end{bmatrix} < 0. \tag{9}$$

Hence, $v^*(t) = \gamma^{-2}D^T(\theta)Zx(t)$ in (5) is obtained.

Furthermore, inequality (9) can be re-written in the following format:

$$\sum_{k=1}^{M} \alpha_k^2 \boldsymbol{M}_k^0(Z,U) + \sum_{k=1}^{M-1} \sum_{\ell=k+1}^{M} \alpha_k \alpha_\ell \boldsymbol{M}_{k\ell}^0(Z,U) < 0. \tag{10}$$

Thus, by using a similar technique to that given in Apkarian *et al.* [1995] and Ku and Wu [2015], the equivalence of $\boldsymbol{M}(Z,U,\theta) < 0$ and (3) can be proved. Therefore, if inequality (3) holds,

$$\mathscr{L}V_v(x,t) + \|z(t)\|^2 - \gamma^2 \|v(t)\|^2 \leq 0. \tag{11}$$

That is, if $v(t) \equiv 0$, $\mathscr{L}V_v(x,t) \leq 0$ holds. Therefore, by using the technique similar to the proof in [Cao and Lam, 2000], the following inequality can be obtained:

$$\lim_{t_f \to \infty} \mathbb{E}\left[\int_0^{t_f} x^T(t,\phi)x(t,\phi)dt\right] \leq x^T(0)\Lambda x(0), \tag{12}$$

where

$$\Lambda = \frac{\lambda_{\max}[Z] + \varepsilon^2 h \lambda_{\max}[U]}{\nu \lambda_{\min}[Z]} I_n, \quad \nu := \min_{\theta} \left\{ \frac{\lambda_{\min}[-\boldsymbol{M}(Z,U,\theta)]}{\lambda_{\max}[Z] + \varepsilon^2 h \lambda_{\max}[U]} \right\}.$$

As a result, the stochastic LPV time-delay system (1) is stochastically mean-square stable. Furthermore, using Dynkin's formula, we obtain

$$\begin{aligned} J_v(x(0),t_f) &\leq -\mathbb{E}\left[\int_0^{t_f} \mathscr{L}V_v(x,t)dt\right] = \mathbb{E}[V_v(x(0),0)] - \mathbb{E}[V_v(x(t_f),t_f)] \\ &\leq \mathcal{F}(Z,U), \end{aligned} \tag{13}$$

where

$$J_v(x(0),t_f) := \mathbb{E}\left[\int_0^{t_f} \|z(t)\|^2 - \gamma^2 \|v(t)\|^2 dt\right].$$

Considering that the stochastic LPV time-delay system is stochastically mean-square stable when t_f approaches infinity, inequality (4) holds. □

Next, the stochastical mean-square stability and the quadratic cost bound for stochastic LPV time-delay system (1) are investigated. Consider the following stochastic LPV time-delay system:

$$dx(t) = [A(\theta)x(t) + A_h(\theta)x(t-h)]dt + A_p(\theta)x(t)dw(t), \tag{14a}$$

$$x(t) = \phi(t), \quad t \in [-h, 0]. \tag{14b}$$

Define the quadratic cost function as

$$J(x^0) = \mathbb{E}\left[\int_0^{\infty} x^T(t)Q(\theta)x(t)dt\right], \tag{15}$$

where

$$Q(\theta) = Q^T(\theta) = \sum_{k=1}^{M} \alpha_k(t) Q_k > 0.$$

Theorem 2. *Assume that there exist matrices* $P = P^T > 0$ *and* $S = S^T > 0$ *satisfying the following LMIs*:

$$\boldsymbol{L}_k^0(P, S) < 0, \tag{16a}$$

$$\boldsymbol{L}_{k\ell}^0(P, S) < 0, \tag{16b}$$

where

$$\begin{aligned}
\boldsymbol{L}_k^0(P, S) &:= \begin{bmatrix} \Upsilon_k^0 & PA_{hk} & A_{pk}^T P \\ A_{hk}^T P & -S & 0 \\ PA_{pk} & 0 & -P \end{bmatrix}, \\
\boldsymbol{L}_{k\ell}^0(P, S) &:= \begin{bmatrix} \Upsilon_{k\ell}^0 & PA_{hk\ell} & A_{pk\ell}^T P \\ A_{hk\ell}^T P & -2S & 0 \\ PA_{pk\ell} & 0 & -2P \end{bmatrix}, \\
k < \ell, \quad k &= 1, \ldots, M, \\
\Upsilon_k^0 &= \Upsilon_k^0(P, S) := PA_k + A_k^T P + S + Q_k, \\
\Upsilon_{k\ell}^0 &= \Upsilon_{k\ell}^0(P, S) := PA_{k\ell} + A_{k\ell}^T P + S + Q_k + Q_\ell.
\end{aligned}$$

Then, we have

$$J(x^0) < \mathcal{F}(P, S), \tag{17}$$

where

$$\mathcal{F}(P, S) := \mathbb{E}[x^T(0)Px(0)] + \mathbb{E}\left[\int_{-h}^{0} \phi^T(s) S \phi(s) ds\right].$$

Proof. We introduce the following parameter independent Lyapunov–Krasovskii function

$$V_u(x, t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Sx(s)ds, \tag{18}$$

where $P = P^T > 0$ and $S = S^T > 0$.

Following similar steps as in the previous discussion, suppose that there exist $P > 0$ and S such that

$$\mathscr{L}V_u(x, t) + x^T(t)Q(\theta)x(t) = \xi^T(t)\boldsymbol{L}(P, S, \theta)\xi(t), \tag{19}$$

where

$$\begin{aligned}
\mathscr{L}V_u(x, t) &= x^T(t)Sx(t) - x^T(t-h)Sx(t-h) \\
&\quad + 2x^T(t)P[A(\theta)x(t) + A_h(\theta)x(t-h)] + x^T(t)A_p^T(\theta)PA_p(\theta)x(t), \\
\boldsymbol{L}(P, S, \theta) &= \begin{bmatrix} \Psi(P, S, \theta) & PA_h(\theta) \\ A_h^T(\theta)P & -S \end{bmatrix}, \\
\Psi(P, S, \theta) &:= PA(\theta) + A^T(\theta)P + S + A_p^T(\theta)PA_p(\theta) + Q(\theta).
\end{aligned}$$

Then, $\boldsymbol{L}(P, S, \theta) < 0$ holds due to $Q(\theta) > 0$. Therefore, the stochastic LPV time-delay system in (14) is mean-square stable. Furthermore, using the result of the stochastic stability of (14), we have

$$J(x^0) < V_u(x, 0) = \mathcal{F}(P, S). \tag{20}$$

By re-writing inequality $\boldsymbol{L}(P, S, \theta) < 0$,

$$\sum_{k=1}^{M} \alpha_k^2 \boldsymbol{L}_k^0(P, S) + \sum_{k=1}^{M-1} \sum_{\ell=k+1}^{M} \alpha_k \alpha_\ell \boldsymbol{L}_{k\ell}^0(P, S) < 0. \tag{21}$$

Thus, if both inequalities in (16) are satisfied, the robust stability and the quadratic cost bound (17) are obtained. □

3. Problem Formulation

Consider a stochastic LPV time-delay system with multiple decision makers defined by

$$dx(t) = \left[A(\theta)x(t) + A_h(\theta)x(t-h) + \sum_{j=1}^{N} B_j(\theta)u_j(t) \right.$$
$$\left. + D(\theta)v(t) \right] dt + A_p(\theta)x(t)dw(t), \tag{22a}$$

$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{22b}$$

$$z(t) = \begin{bmatrix} E(\theta)x(t) \\ H_1 u_1(t) \\ \vdots \\ H_N u_N(t) \end{bmatrix}, \tag{22c}$$

where $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$, denotes the ith decision maker's control input. The other variables are defined by stochastic equation (1). The coefficient matrices $B_i(\theta)$, $i = 1, \ldots, N$ in (22) can be expressed as

$$B_i(\theta) = \sum_{k=1}^{M} \alpha_k(t) B_{ik}. \tag{23}$$

Furthermore, note that H_i does not depend on a time-varying parameter, because the controlled output can be chosen by the controller designer. Hence, without loss of generality, it can be assumed that H_i is a constant matrix.

Assumption 1. $H_i^T H_i = I_{m_i}$, $i = 1, \ldots, N$, $H_i \in \mathbb{R}^{m_{q_i} \times m_i}$. Furthermore, without loss of generality, to remove the dependence on $x(0)$, assume that $x(0)$ is a zero mean random variable satisfying $\mathbb{E}[x(0)x^T(0)] = I_n$.

The cost performance functions are defined by

$$J_v(u_1, \ldots, u_N, v, x^0) = \mathbb{E}\left[\int_0^\infty [\gamma^2\|v(t)\|^2 - \|z(t)\|^2]dt\right], \tag{24a}$$

$$J_i(u_i, v, x^0) = \mathbb{E}\left[\int_0^\infty \{x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t)\}dt\right], \tag{24b}$$

where $Q_i = Q_i^T > 0$, $R_i = R_i^T > 0$, $i = 1, \ldots, N$.

The problem of H_∞ constrained Pareto strategy [Engwerda, 2005] for stochastic LPV system (22) is stated as follows.

Problem 1. For any given positive parameter γ, $v(t) = v(t) \in L_F^2([0,\infty), \mathbb{R}^{m_v})$, find a Pareto suboptimal state feedback memoryless strategy set

$$u_i(t) = u_i^*(t) = K_i(\theta)x(t) = \sum_{k=1}^{M} \alpha_k K_{ik} x(t) \tag{25}$$

such that

(i) $u_i(t) = u_i^*(t)$, $i = 1, \ldots, N$, make stochastic system LPV (22) stochastically mean-square stable when $v(t) = 0$ and the following inequality holds:

$$\|z\|_2^2 \le \gamma^2\|v\|_2^2 + \mathcal{F}(Z, U); \tag{26}$$

(ii) When $v(t) = v^*(t) = F_\gamma^* x(t)$ is applied, consider a weighted sum of the cost function of the followers, given by

$$\begin{aligned} J_\rho(u_1, \ldots, u_N, v^*, x^0) &:= \sum_{i=1}^{N} \rho_k J_i(u_i, v^*, x^0), \\ \sum_{i=1}^{N} \rho_i = 1, \quad 0 < \rho_i &< 1, \quad i = 1, \ldots, N. \end{aligned} \tag{27}$$

A Pareto suboptimal strategy set $(u_1, \ldots, u_N)$ minimizes the upper bound defined by the following cost function:

$$\bar{J}_\rho(u_1, \ldots, u_N, v^*, x^0) = \sup_{\theta(t)} J_\rho(u_1, \ldots, u_N, v^*, x^0). \tag{28}$$

In the following section, the existence conditions of the H_∞ constrained Pareto suboptimal strategy set are established in terms of the preliminary results.

4. Main Results

The main result related to the quadratic cost bound under the H_∞ constraint condition is given in the following.

Theorem 3. *Consider stochastic LPV time-delay system* (22) *with multiple decision makers* $u_i(t) = K_i(\theta)x(t)$, $i = 1, \ldots, N$, *and deterministic disturbance*

$v(t) = F_\gamma(\theta)x(t)$. *Given attenuation performance level* γ, *assume that there exists a strategy set for the real symmetric matrices* $X = X^T > 0$, $W = W^T > 0$, $Z = Z^T > 0$, $U = U^T > 0$ *and* Y_k *such that the following cross-coupled BMIs are satisfied*:

$$\boldsymbol{L}_k(X, W, Y_k, F_{\gamma k}) < 0, \tag{29a}$$

$$\boldsymbol{L}_{k\ell}(X, W, Y_k, Y_\ell, F_{\gamma k}, F_{\gamma \ell}) < 0, \tag{29b}$$

$$\boldsymbol{M}_k(Z, U, K_k) < 0, \tag{29c}$$

$$\boldsymbol{M}_{k\ell}(Z, U, K_k, K_\ell) < 0, \tag{29d}$$

where $k < \ell$, $k = 1, \ldots, M$,

$$\boldsymbol{L}_k(X, W, Y_k, F_{\gamma k}) = \begin{bmatrix} \Upsilon_k & XA_{pk}^T & X & X & Y_k^T \\ A_{pk}X & -X & 0 & 0 & 0 \\ X & 0 & -Q_\rho^{-1} & 0 & 0 \\ X & 0 & 0 & -W & 0 \\ Y_k & 0 & 0 & 0 & -R_\rho^{-1} \end{bmatrix},$$

$$\boldsymbol{L}_{k\ell}(X, W, Y_k, Y_\ell, F_{\gamma k}, F_{\gamma \ell}) := \begin{bmatrix} \Upsilon_{k\ell} & XA_{pk\ell}^T & X & X & Y_{k\ell}^T \\ A_{pk\ell}X & -2X & 0 & 0 & 0 \\ X & 0 & -Q_\rho^{-1} & 0 & 0 \\ X & 0 & 0 & -W & 0 \\ Y_{k\ell} & 0 & 0 & 0 & -2R_\rho^{-1} \end{bmatrix},$$

$$\boldsymbol{M}_k(Z, U, K_k) := \begin{bmatrix} \Xi_k & ZA_{hk} & ZD_k & A_{pk}^T Z & E_{Kk}^T \\ A_{hk}^T Z & -U & 0 & 0 & 0 \\ D_k^T Z & 0 & -\gamma^2 I_{n_v} & 0 & 0 \\ ZA_{pk} & 0 & 0 & -Z & 0 \\ E_{Kk} & 0 & 0 & 0 & -I_{n_z} \end{bmatrix},$$

$$\boldsymbol{M}_{k\ell}(Z, U, K_k, K_\ell) := \begin{bmatrix} \Xi_{k\ell} & ZA_{hk\ell} & ZD_{k\ell} & A_{pk\ell}^T Z & E_{Kk\ell}^T \\ A_{hk\ell}^T Z & -2U & 0 & 0 & 0 \\ D_{k\ell}^T Z & 0 & -2\gamma^2 I_{n_v} & 0 & 0 \\ ZA_{pk\ell} & 0 & 0 & -2Z & 0 \\ E_{Kk\ell} & 0 & 0 & 0 & -2I_{n_z} \end{bmatrix},$$

$$\begin{aligned} \Upsilon_k &= \Upsilon_k(X, W, Y_k, F_{\gamma k}) := X(A_k + D_k F_{\gamma k})^T \\ &\quad + (A_k + D_k F_{\gamma k})X + B_{ck}Y_k + Y_k^T B_{ck}^T + A_{hk} W A_{hk}^T, \end{aligned}$$

$$
\begin{aligned}
\Upsilon_{k\ell} &= \Upsilon_{k\ell}(X, W, Y_k, Y_\ell, F_{\gamma k}, F_{\gamma\ell}) \\
&:= X(A_{k\ell} + D_k F_{\gamma\ell} + D_\ell F_{\gamma k})^T + (A_{k\ell} + D_k F_{\gamma\ell} \\
&\quad + D_\ell F_{\gamma k})X + B_{ck} Y_\ell + B_{c\ell} Y_k + Y_k^T B_{c\ell}^T \\
&\quad + Y_\ell^T B_{ck}^T + \frac{1}{2} A_{hk\ell} W A_{hk\ell}^T, \\
\Xi_k &= \Xi_k(Z, U, K_k) := Z(A_k + B_{ck} K_k) \\
&\quad + (A_k + B_{ck} K_k)^T Z + U, \\
\Xi_{k\ell} &= \Xi_{k\ell}(Z, U, K_k, K_\ell) := Z(A_{k\ell} + \Theta_{ck\ell}) \\
&\quad + (A_{k\ell} + \Theta_{ck\ell})^T Z + U, \\
Y_{k\ell} &:= Y_k + Y_\ell, \ K_{k\ell} := K_k + K_\ell, \\
E_{Kk\ell} &:= E_{Kk} + E_{K\ell}, \\
B_{ck} &= [B_{1k} \quad \cdots \quad B_{Nk}], \\
K_k &:= \begin{bmatrix} K_{1k} \\ \vdots \\ K_{Nk} \end{bmatrix} = Y_k X^{-1}, \quad E_{Kk} := \begin{bmatrix} E_k \\ L_1 K_{1k} \\ \vdots \\ L_N K_{Nk} \end{bmatrix}, \\
F_{\gamma k} &= \gamma^{-2} D_k^T Z, \ Q_\rho := \sum_{j=1}^{N} \rho_j Q_j, \\
R_\rho &:= \mathbf{block\ diag}(\rho_1 R_1 \quad \cdots \quad \rho_N R_N).
\end{aligned}
$$

Then, the H_∞ constrained Pareto suboptimal strategy set under consideration is given by

$$
u_i^*(t) = K_i(\theta)x(t) = \sum_{k=1}^{M} \alpha_k K_{ik} x(t). \tag{30}
$$

Furthermore, the optimal cost bounds are given by

$$
J_\rho(u_1, \ldots, u_N, v^*, x^0) < \bar{J}_\rho(u_1, \ldots, u_N, v^*, x^0) = \mathcal{F}(P, S), \tag{31}
$$

with the worst case disturbance

$$
v(t) = v^*(t) = F_\gamma(\theta)x(t) = \gamma^{-2} D^T(\theta) Z x(t). \tag{32}
$$

Proof. First, the H_∞ constraint condition is derived. By applying Pareto suboptimal strategy set (30) to the original stochastic LPV time-delay system in (22), we have the following closed-loop stochastic LPV time-delay system:

$$dx(t) = \left[\left(A(\theta) + \sum_{j=1}^{N} B_j(\theta)K_j(\theta)\right)x(t) + A_h(\theta)x(t-h) \right.$$
$$\left. + D(\theta)v(t)\right]dt + A_p(\theta)x(t)dw(t), \tag{33a}$$
$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{33b}$$
$$z(t) = E_K(\theta)x(t). \tag{33c}$$

By termwise comparison between (1) and (33), we have

$$B(\theta)K(\theta) \leftarrow \sum_{j=1}^{N} B_j(\theta)K_j(\theta) = B_c(\theta)K_c(\theta)$$
$$= \sum_{k=1}^{M} \alpha_k^2 B_{ck}K_k + \sum_{k=1}^{M-1}\sum_{\ell=k+1}^{M} \alpha_k\alpha_\ell(B_{ck}K_\ell + B_{c\ell}K_k),$$
$$E(\theta) \leftarrow E_K(\theta) = \sum_{k=1}^{M} \alpha_k E_{Kk}.$$

Thus, by applying Theorem 1 to this problem, the conditions (29c) and (29d) can be obtained.

Second, the existence condition of the Pareto suboptimal strategy set is derived. Consider the stochastic LPV time-delay system with the following cost function:

$$J_\rho(K_1(\theta)x, \ldots, K_M(\theta)x, v^*, x^0)$$
$$= \mathbb{E}\left[\int_0^\infty x^T(t)[Q_\rho + K_c(\theta)^T R_\rho K_c(\theta)]x(t)dt\right], \tag{34}$$

such that

$$dx(t) = [[A(\theta) + B_c(\theta)K_c(\theta) + D(\theta)F_\gamma(\theta)]x(t)$$
$$+ A_h(\theta)x(t-h)]dt + A_p(\theta)x(t)dw(t), \tag{35a}$$
$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \sum_{k=1}^{M} \alpha_k \begin{bmatrix} K_{1k} \\ \vdots \\ K_{Nk} \end{bmatrix} x(t) = \sum_{k=1}^{M} \alpha_k K_k x(t). \tag{35b}$$

Following steps similar to those in the previous problem and by termwise comparison between (14), (15) and (35), (34) with

$$A(\theta) \leftarrow A(\theta) + B_c(\theta)K_c(\theta) + D(\theta)F_\gamma(\theta),$$
$$Q \leftarrow Q_\rho + K_c(\theta)^T R_\rho K_c(\theta),$$

we have the following matrix inequalities using Theorem 2:

$$\bar{\boldsymbol{L}}_k(P, S, K_k, F_{\gamma k}) < 0, \tag{36a}$$

$$\bar{\boldsymbol{L}}_{k\ell}(P, S, K_k, K_\ell, F_{\gamma k}, F_{\gamma \ell}) < 0, \tag{36b}$$

where $k < \ell$, $k = 1, \ldots, M$,

$$\begin{aligned}
\bar{\boldsymbol{L}}_k(P, S, K_k, F_{\gamma k}) &= \begin{bmatrix} \Delta_k & PA_{hk} & A_{pk}^T P \\ A_{hk}^T P & -S & 0 \\ PA_{pk} & 0 & -P \end{bmatrix}, \\
\bar{\boldsymbol{L}}_{k\ell}(P, S, K_k, K_\ell, F_{\gamma k}, F_{\gamma \ell}) &:= \begin{bmatrix} \Delta_{k\ell} & PA_{hk\ell} & A_{pk\ell}^T P \\ A_{hk\ell}^T P & -2S & 0 \\ PA_{pk\ell} & 0 & -2P \end{bmatrix}, \\
\Delta_k &= \Delta_k(P, S, K_k, F_{\gamma k}) \\
&:= P(A_k + B_{ck} K_k + D_k F_{\gamma k}) \\
&\quad + (A_k + B_{ck} K_k + D_k F_{\gamma k})^T P \\
&\quad + S + Q_\rho + K_k^T R_\rho K_k, \\
\Delta_{k\ell} &= \Delta_{k\ell}(P, S, K_k, K_\ell, F_{\gamma k}, F_{\gamma \ell}) \\
&:= P(A_{k\ell} + B_{ck} K_\ell + D_k F_{\gamma \ell} + B_{c\ell} K_k + D_\ell F_{\gamma k}) \\
&\quad + (A_{k\ell} + B_{ck} K_\ell + D_k F_{\gamma \ell} + B_{c\ell} K_k + D_\ell F_{\gamma k})^T P \\
&\quad + S + Q_\rho + K_{k\ell}^T R_\rho K_{k\ell}.
\end{aligned}$$

Applying the Schur complement lemma to inequalities (36) and multiplying the following matrix

$$\textbf{block diag}(P^{-1} \quad I_n \quad I_n \quad I_n \quad I_m), \quad m := \sum_{i=1}^{N} m_i$$

on both sides, LMIs (29a) and (29b) can be obtained. In addition, the quadratic cost bound of (34) can be derived as (31). □

It should be noted that the optimization problem related to the upper bound of the cost $\bar{J}_\rho(u_1, \ldots, u_N, v^*, x^0)$ in (31) is tackled in the following subsection in the proposed iterative algorithm.

4.1. *Iterative Algorithm*

In order to obtain the H_∞ constrained Pareto suboptimal strategy set of (30), we need to solve the cross-coupled BMIs (29). In particular, the following optimization should be solved:

$$\min_{X,W,Y_1,\ldots,Y_N} \bar{J}_\rho(u_1,\ldots,u_N,v^*,x^0)$$
$$= \min_{X,W,Y_1,\ldots,Y_N} (\mathbf{Trace}[X^{-1}] + \mathbf{Trace}[LL^TW^{-1}]), \tag{37a}$$
$$\text{s.t. } (29a), \quad (29b), \tag{37b}$$

where $LL^T = \int_{-h}^{0} \phi(s)\phi^T(s)ds$.

In the following, an iterative algorithm including the optimization problem (37) by means of combining LMIs with the viscosity iterative scheme Xu [2004] is established:

Step 1. Set the initial values: choose an appropriate $F_{\gamma k}^{(0)}$;

Step 2. Solve the following optimization problem based on the LMI for $X^{(n+1)}$, $W^{(n+1)}$ $Y_k^{(n+1)}$:

$$\min_{\boldsymbol{x}^{(n+1)}} \mathbf{Tr}[\Gamma^{(n+1)}] + \mathbf{Tr}[\tau^{(n+1)}], \tag{38a}$$
$$\boldsymbol{x}^{(n+1)} := (X^{(n+1)}, W^{(n+1)}Y_1^{(n+1)},\ldots,Y_M^{(n+1)},\Gamma^{(n+1)},\tau^{(n+1)}),$$
$$\text{s.t. } \boldsymbol{L}_k(X^{(n+1)}, W^{(n+1)}, Y_k^{(n+1)}, F_{\gamma k}^{(n)}) < 0, \tag{38b}$$
$$\boldsymbol{L}_{k\ell}(X^{(n+1)}, W^{(n+1)}, Y_k^{(n+1)}, Y_\ell^{(n+1)}, F_{\gamma k}^{(n)}, F_{\gamma \ell}^{(n)}) < 0, \tag{38c}$$
$$\begin{bmatrix} -\Gamma^{(n+1)} & I_n \\ I_n & -X^{(n+1)} \end{bmatrix} < 0, \tag{38d}$$
$$\begin{bmatrix} -\tau^{(n+1)} & L^T \\ L & -W^{(n+1)} \end{bmatrix} < 0. \tag{38e}$$

Step 3. Compute the strategy set:

$$K_k^{(n+1)} = Y_k^{(n+1)}[X^{(n+1)}]^{-1}, \quad k = 1,\ldots,M. \tag{39}$$

Step 4. Solve the following optimization problem for $Z^{(n+1)}$ and $U^{(n+1)}$:

$$\min_{Z^{(n+1)},U^{(n+1)}} \mathbf{Tr}[Z^{(n+1)}] + \mathbf{Tr}[U^{(n+1)}], \text{ s.t.,} \tag{40a}$$
$$\boldsymbol{M}_k(Z^{(n+1)}, U^{(n+1)}, K_k^{(n+1)}) < 0, \tag{40b}$$
$$\boldsymbol{M}_{k\ell}(Z^{(n+1)}, U^{(n+1)}, K_k^{(n+1)}, K_\ell^{(n+1)}) < 0, \tag{40c}$$

where

$$F_{\gamma k}^{(n+1)} := \gamma^{-2}D_k^T Z^{(n+1)}, \quad E_{Kk}^{(n+1)} := \begin{bmatrix} E_k \\ L_1K_{1k}^{(n+1)}C_1 \\ \vdots \\ L_NK_{Nk}^{(n+1)}C_N \end{bmatrix}.$$

Step 5. For any appropriate fixed value of $\eta \in (0, 1)$, set

$$\boldsymbol{z}^{(n+1)} \leftarrow \eta(1 - \beta_n)\boldsymbol{z}^{(n)} + \beta_n \boldsymbol{z}^{(n+1)}, \tag{41}$$

where $\mathscr{T}$ is the mapping from Steps 2–4 such that

$$\boldsymbol{z}^{(n+1)} = \mathscr{T}(\boldsymbol{z}^{(n)}),$$

$$\boldsymbol{z}^{(n)} := \left[\mathrm{vec}[X^{(n)}]\ \mathrm{vec}[W^{(n)}]\ \mathrm{vec}[Y_1^{(n)}]\ \cdots\ \mathrm{vec}[Y_M^{(n)}]\ \mathrm{vec}[Z^{(n)}]\ \mathrm{vec}[U^{(n)}]\right].$$

Furthermore, parameter $\{\beta_n\} \in [0, 1]$ satisfies the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \tag{42a}$$

$$\sum_{n=0}^{\infty} \beta_n = \infty, \tag{42b}$$

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty. \tag{42c}$$

Step 6. If the iterative algorithm consisting of Steps 2–5 converges, we have obtained solutions for the cross-coupled BMIs (29); otherwise, if the number of iterations reaches a preset threshold, declare that there is no strategy set. Stop.

The fact theorem indicates the norm-convergence of the proposed iterative scheme that is known as the strong convergence property that has been proved in [Xu, 2004].

Fact 1. If $\mathscr{T}$ is a monotone nonexpansive mapping with a fixed point, then the viscosity iterative scheme $\{\boldsymbol{z}^{(n)}\}$ converges strongly to a fixed point in uniformly smooth Banach space.

5. Numerical Examples

In order to show the effectiveness of the proposed H_∞ constrained Pareto suboptimal strategy, two numerical examples are presented.

5.1. *Academic example*

Consider the following stochastic LPV time-delay system with two decision makers, modified from [Wu and Grigoriadis, 2001]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 + 0.2\sin t \\ -2 & -3 + 0.1\sin t \end{bmatrix} x(t) + \begin{bmatrix} 0.2\sin t & 0.1 \\ -0.2 + 0.1\sin t & -0.3 \end{bmatrix} x(t-h)$$

$$+ \begin{bmatrix} 0.2\sin t \\ 0.1 + 0.1\sin t \end{bmatrix} u_1(t) + \begin{bmatrix} 0.1 + 0.1\sin t \\ 0.2\sin t \end{bmatrix} u_2(t) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} v(t),$$

$$z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(t).$$

Note that the stochastic LPV with multiple control inputs is considered, different from that in Wu and Grigoriadis [2001].

It is assumed that the state-dependent noise $A_p(\theta)x(t)dw(t)$ has 15% perturbation based on state matrix $A(\theta)$. Using the fact that $\sin t = \cos^2(\pi/4 - t/2) - \sin^2(\pi/4 - t/2)$, the system matrices of (22) are given by

$$A_1 = \begin{bmatrix} 0 & 1.2 \\ -2 & -2.9 \end{bmatrix}, \quad A_{h1} = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & -0.3 \end{bmatrix}, \quad A_{p1} = 0.15A_1,$$

$$A_2 = \begin{bmatrix} 0 & 0.8 \\ -2 & -3.1 \end{bmatrix}, \quad A_{h2} = \begin{bmatrix} -0.2 & 0.1 \\ -0.3 & -0.3 \end{bmatrix}, \quad A_{p2} = 0.15A_2,$$

$$B_{11} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},$$

$$D_1 = D_2 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad h = 0.1, \quad \phi(t) = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad -h \le t \le 0,$$

$$Q_1 = \mathrm{diag}(0.9 \quad 1.9), \quad Q_2 = \mathrm{diag}(1.2 \quad 1.2),$$

$$R_1 = 0.9, \quad R_2 = 2.4, \quad \rho_1 = \rho_2 = 0.5.$$

The disturbance attenuation level is set as $\gamma = 3.1$. The cross-coupled BMIs in (29) are solved by using the proposed viscosity iterative scheme. The computed strategy set in (30) and the worst case disturbance in (32) with the related solution matrices are given by

$$P = X^{-1} = \begin{bmatrix} 2.5632 & 4.5954 \times 10^{-1} \\ 4.5954 \times 10^{-1} & 5.2412 \times 10^{-1} \end{bmatrix},$$

$$S = W^{-1} = \begin{bmatrix} 4.3870 \times 10^{-1} & -8.5127 \times 10^{-2} \\ -8.5127 \times 10^{-2} & 1.4887 \times 10^{-1} \end{bmatrix},$$

$$K_{11} = Y_{11}X^{-1} = [-4.8206 \times 10^{-1} \quad -8.9643 \times 10^{-1}],$$

$$K_{21} = Y_{21}X^{-1} = [1.0418 \times 10^{-1} \quad 2.4881 \times 10^{-1}],$$

$$K_{12} = Y_{12}X^{-1} = [6.0283 \times 10^{-1} \quad -6.1613 \times 10^{-1}],$$

$$K_{22} = Y_{22}X^{-1} = [-3.0265 \times 10^{-1} \quad 1.4370 \times 10^{-1}],$$

$$Z = \begin{bmatrix} 1.6635 & 2.3750 \times 10^{-1} \\ 2.3750 \times 10^{-1} & 4.6005 \times 10^{-1} \end{bmatrix},$$

$$U = \begin{bmatrix} 2.3961 \times 10^{-1} & -7.2524 \times 10^{-2} \\ -7.2524 \times 10^{-2} & 1.3598 \times 10^{-1} \end{bmatrix},$$

$$F_{\gamma 1} = F_{\gamma 2} = [3.9564 \times 10^{-2} \quad 1.4517 \times 10^{-2}].$$

Next, the proposed viscosity iterative scheme to obtain the converged solutions and the strategies is verified. The initial value is set as $F_{\gamma k}^{(0)} = 2\gamma^{-2} D_k^T I_n$ for $k = 1, 2$. In Step 5 of the proposed algorithm, the value of η is set to 0.99 and $\beta_n = \frac{1}{n}$. The proposed algorithm converges after 691 iterations with an accuracy of $\psi^{(n)} := \|\boldsymbol{z}^{(n+1)} - \boldsymbol{z}^{(n)}\| < 10^{-6}$.

Finally, the robust stability of the stochastic LPV time-delay system is confirmed. Figure 1 shows that all the states attain the mean-square stability, in the presence of state delay and multiple decision makers.

5.2. *Williams–Otto process*

Second, we discuss a practical numerical example based on the Williams–Otto process Williams and Otto [1960]; Ross [1971]; Ahmadizadeh *et al.* [2014], which is well known as a nonisothermal continuous stirred-tank reactor (CSTR). It should be noted that it is widely known as a typical chemical process adopted in the control engineering literature. In this example, stochastic LPV delay system representations of the nonlinear model of the CSTR are developed to adjust the system nonlinearities in the LPV scheduling variables Puna and Bakošovà M [2007]; Rahme *et al.* [2015]. Consider the linearized delay stochastic system of the CSTR was obtained in the following form:

$$
\begin{aligned}
dx(t) &= \Bigg[A(\theta)x(t) + A_h(\theta)x(t-h) \\
&\qquad + \sum_{j=1}^{2} B_j(\theta)u_j(t) + D(\theta)v(t)\Bigg] dt + A_p(\theta)x(t)dw(t), \\
x(t) &= \phi(t), \quad t \in [-1, 0], \\
z(t) &= \begin{bmatrix} E(\theta)x(t) \\ H_1 u_1(t) \\ H_2 u_2(t) \end{bmatrix},
\end{aligned}
$$

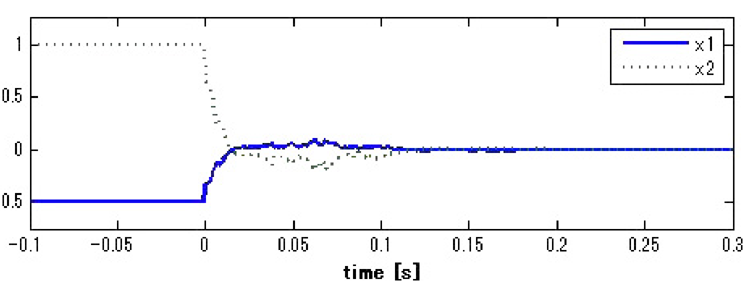

Fig. 1. State trajectories.

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad \phi(t) = \begin{bmatrix} 0.4 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}.$$

Due to the variation of the reaction rate in nonlinear dynamics, the linearized mathematical models $M = 2$ are assumed and suppose that 5% of the perturbation exists from the nominal value of the state matrix of A. Therefore, the coefficient matrices are defined as follows:

$$A = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix},$$

$$A_1 = A + 0.05A, \quad A_2 = A - 0.05A,$$

$$A_{hk} = \mathbf{diag}(1.92 \quad 1.92 \quad 1.87 \quad 0.724), \quad A_{pk} = 0.05A_k,$$

$$B_{1k} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{2k} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad D_k = \begin{bmatrix} 0.1 \\ -0.2 \\ -1 \\ 0.5 \end{bmatrix},$$

$$E_k = [0.1 \quad 0.1 \quad 0 \quad 0], \quad H_i = [1], \quad i = 1, 2, \ \ k = 1, 2.$$

These systems represent vertices of the uncertain polytopic system. On the other hand, we also consider that 5% of the state coefficient can be represented by a Wiener process due to stochastic perturbations.

The weight matrices of cost functions and the ρ_i are given by

$$Q_1 = 4I_4, \quad Q_2 = 2I_4, \quad R_1 = 1, \quad R_2 = 2, \quad \rho = 1 = \rho_2 = 0.5.$$

Next, we select $\gamma = 3$. Using the proposed iterative algorithms in Sec. 4.1 with the initial conditions

$$F_{\gamma k}^{(0)} = \gamma^{-2} D_k^T I_4, \quad k = 1, \ \ 2,$$

we obtain the following H_∞ constrained Pareto suboptimal strategy set in (30):

$$K_{11} = [-2.1731 \quad 8.4985 \times 10^{-1} \quad -4.3998 \times 10^{-1} \quad -1.2814 \times 10^{-1}],$$

$$K_{21} = [4.1397 \times 10^{-1} \quad -5.6098 \times 10^{-1} \quad 1.5535 \times 10^{-1} \quad -1.4498 \times 10^{-1}],$$

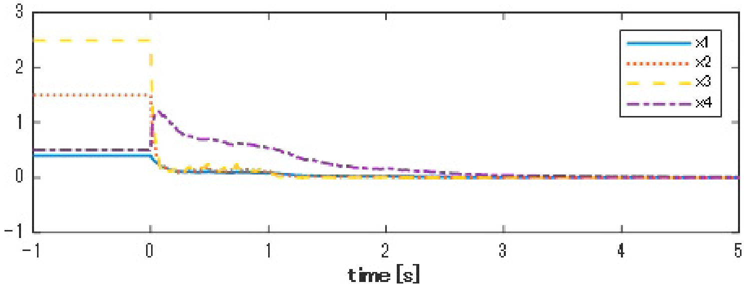

Fig. 2. State trajectories of the Williams–Otto model.

$$K_{12} = [-2.1731 \quad 8.4985 \times 10^{-1} \quad -4.3998 \times 10^{-1} \quad -1.2814 \times 10^{-1}],$$

$$K_{22} = [2.5496 \times 10^{-1} \quad -3.4729 \times 10^{-1} \quad 9.5814 \times 10^{-2} \quad -9.0458 \times 10^{-2}],$$

$$F_{\gamma 1} = F_{\gamma 2} = [-1.0674 \times 10^{-3} \quad 7.0751 \times 10^{-3} \quad 1.0569 \times 10^{-5} \quad 8.0742 \times 10^{-3}].$$

It should be noted that the proposed iterative algorithm converges after 52 iterations with an accuracy of 10^{-4}, and the strategy set was computed.

Second, Fig. 2 shows how the system states with time. In the case that the deterministic time-varying uncertainty is $\alpha_1(t) = e^{-t}$, $\alpha_2(t)(t) = 1 - e^{-t}$ as a special case, it is observed that all the states are stable even though the delay and the stochastic noise exist.

6. Conclusion

In this paper, we have studied an H_∞ constrained Pareto suboptimal strategy for stochastic LPV systems with constant state delay. The challenge is to achieve stochastical mean-square stability even when several unsatisfactory factors exist such as the time delay and disturbances. As a main contribution, the existence conditions of the H_∞ constrained Pareto suboptimal strategy set are derived via cross-coupled BMIs. In addition, a computational framework based on isolated LMIs using the viscosity iterative scheme Xu [2004] is proposed to avoid directly processing BMIs. Finally, two numerical examples are solved to demonstrate the effectiveness and convergence of the proposed algorithm.

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