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Robust Nash Static Output Feedback Strategy for Uncertain Markov Jump Delay Stochastic Systems

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Abstract— **In this paper, we propose a robust Nash strategy for a class of uncertain Markov jump delay stochastic systems (UMJDSSs) via static output feedback (SOF). After establishing the extended bounded real lemma for UMJDSS, the conditions for the existence of a robust Nash strategy set are determined by means of cross coupled stochastic matrix inequalities (CCSMIs). In order to solve the SOF problem, an heuristic algorithm is developed based on the algebraic equations and the linear matrix inequalities (LMIs). In particular, it is shown that robust convergence is guaranteed under a new convergence condition. Finally, a practical numerical example based on the congestion control for active queue management is provided to demonstrate the reliability and usefulness of the proposed design scheme.**

I. INTRODUCTION

The congestion control based on the transmission control protocol (TCP) for active queue management (AQM) is an important operation in communication networks [1]. TCP is known as a representative networking protocol in the Internet. The TCP congestion window size is increased if there is no congestion; otherwise, the window size is reduced drastically. In this way, the TCP avoids congestion by controlling the window size via the feedback information to the transmitting side that the data has been sent to the receiving side. On the other hand, AQM adjusts the transmission of a large amount of data according to the state of the router, under the assumption that congestion occurring in a specific bottleneck router is known. As a basic mechanism, in order to control congestion, incoming packets to each router are stored in a buffer with finite capacity and the packets are dropped before the buffer become full through AQM. Although this congestion control mechanism attains network stability and integrity to some extent, there are some fundamental issues that should be resolved. For example, the overall performance degradation of an end-to-end data delivery is mainly due to the transitions of transmission modes in a bottleneck router, the transmission and feedback delay, interference and noise. Therefore, it is important to investigate the stability of networks under the communication delay, influence of external disturbances, physical constraints to rely on local information, the robust strategy design and numerical approach for congestion control in the network.

Motivated by the preceding issues, a robust Nash static output feedback (SOF) control problem is investigated in this paper for uncertain Markov jump delay stochastic systems (UMJDSSs) with multiple decision makers. A synthesized use of Markov switching, dynamic game, stochastic control and robust control is required to solve the problem. Up to now, uncertain Markov jump linear stochastic systems (UMJLSSs) are used to describe many practical systems characterized by uncertainties in system matrices and changes in operating points [2]. During the past decade, stochastic control problems and stabilization problems of Markov jump linear stochastic systems (MJLSSs) with delays have been studied [3], [4], [5], [6], [7]. In [8], state feedback Pareto suboptimal control and Nash equilibrium for Markov jump delay stochastic delay systems (MJDSSs) have been investigated. Although these studies are effective in solving the corresponding problems, it is assumed that the considered controllers are implemented using the state feedback information.

On the other hand, although the SOF control problems for Markov jump linear deterministic systems have been studied [9], there exist very few rigorous results on the SOF strategies in dynamic games except for [10], [11], [12]. Although recent advances in the robust SOF Pareto suboptimal for the UMJLSSs have been found in [14], studies on robust SOF Nash strategies for UMJLSSs with input delay remain open. This problem is the most fundamental and challenging problem in dynamic game in spite of the computational difficulty. The contributions of this paper are as follows. First, using the guaranteed cost control technique [13], the condition for the existence of the strategy set are formulated by means of a cross coupled stochastic matrix inequalities (CCSMIs). In comparison with the existing results [14], full state information is not needed to design the proposed strategy set. Different from the previous results for the SOF strategy in [2], [10], [11], [12], deterministic bounded uncertainties are considered by relaxing the conservatism for the restriction of using the strategy set. Second, a computational framework for validating heuristic algorithm is proposed, which uses the Krasnoselskii-Mann (KM) iterative algorithm [15] to obtain a solution set of CCSMIs. Finally, in order to demonstrate the reliability and usefulness of the proposed strategy set and the algorithm, a practical AQM congestion control problem is addressed.

Notation: The notations used in this paper are fairly

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standard: $\mathbb{E}[\ \cdot\ \mid r_t = i]$ stands for the conditional expectation operator with respect to the event $\{r_t = i\}$; $\mathbb{M}^s_{n,m}$ denotes space of all $\boldsymbol{S} = (S(1), ..., S(s))$ with $S(i)$ being $n \times m$ matrix, $i \in \mathcal{D}$, $\mathcal{D} = \{1,\ 2, ..., s\}$. Moreover, the components of $\boldsymbol{S} + \boldsymbol{TU}$ are defined as $\boldsymbol{S} + \boldsymbol{TU} = (S(1) + T(1)U(1), ..., S(s) + T(s)U(s))$; $\mathcal{L}^2_F([0,\ \infty),\ \mathbb{R}^k)$ denotes the space of all measurable functions $u(t,\ \omega)\ :\ [0,\ \infty) \times \Omega \to \mathbb{R}^k$, which are F_t-measurable for every $t \geq 0$, and $\mathbb{E}[\int_0^\infty |u(t)|^2 dt \mid r_0 = i] < \infty$, $i \in \mathcal{D}$; $C([-h, 0]; \mathbb{R}^n)$, $h > 0$ denotes the family of continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-h \leq \theta \leq 0} \|\phi(\theta)\|$; $\lambda_{\max}[\cdot]$ and $\lambda_{\min}[\cdot]$ denote its largest and smallest eigenvalue, respectively.

II. PRELIMINARY RESULTS

Let $w(t)$, $t \geq 0$, be the one-dimensional Wiener process that is defined on a given filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathcal{P})$, and r_t, $t \geq 0$, be a right continuous homogeneous Markov process taking values in a finite state space $\mathcal{D} = \{1,\ 2, ..., s\}$. It is assumed that $\{w(t)\}_{t\geq 0}$ and $\{r_t\}_{t\geq 0}$ are independent stochastic processes. Furthermore, the transition probabilities are given by

$$\mathbf{P}\{r_{t+h} = j \mid r_t = i\} = \begin{cases} \pi_{ij} h + o(h), & \text{if } i \neq j \\ 1 + \pi_{ii} h + o(h), & \text{else} \end{cases} \tag{1}$$

where $h > 0$, $\pi_{ij} \geq 0$, $i \neq j$, $\pi_{ii} = -\sum_{j=1,\ j\neq i}^{s} \pi_{ij}$, $\lim_{h\to 0} o(h)/h = 0$.

Consider the following UMJDSS

$$dx(t) = [A(r_t, t)x(t) + A_h(r_t)x(t-h) + B_v(r_t)v(t)]dt + A_p(r_t, t)x(t)dw(t), \tag{2a}$$

$$x(t) = \phi(t),\ t \in [-h,\ 0], \tag{2b}$$

$$z(t) = H(r_t)x(t), \tag{2c}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $v(t) \in \mathbb{R}^{m_v}$ the external disturbance, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}$ a one-dimensional standard Wiener process defined in the filtered probability space, $h > 0$ the time-delay of the UMJDSSs, and $\phi(t)$ a real-valued initial function. Without loss of generality, it is assumed that, for all $\delta \in [-h,\ 0]$, there exists a scalar $\sigma > 0$ such that $\|x(t+\delta)\| \leq \sigma \|x(t)\|$ [16].

Let $A(r_t, t)$ and $A_p(r_t, t) \in \mathbb{R}^{n\times n}$ be matrices with the following forms:

$$A(r_t, t) = A(r_t) + D(r_t)\Theta(r_t, t)E_a(r_t), \tag{3a}$$

$$A_p(r_t, t) = A_p(r_t) + D_p(r_t)\Theta(r_t, t)E_{pa}(r_t), \tag{3b}$$

where $\Theta^T(r_t, t)\Theta(r_t, t) \leq I_{n_a}$.

In coefficients $A(i)$, $A_h(i)$, $A_p(i)$, $D(i)$, $D_p(i)$, $E_a(i)$, $E_{pa}(i)$ and $B_v(i)$, $i \in \mathcal{D}$, these matrices are constant; $\Theta(r_t, t) \in \mathbb{R}^{n_p \times n_a}$ is unknown real matrix representing uncertainties [14].

First, the related definition and lemmas are introduced.

Definition 1: [16], [17] The UMJDSS is said to be stochastically stable if, when $v(t) \equiv 0$, for all finite $\phi(t) \in \mathbb{R}^n$ defined on $[-h,\ 0]$ and initial mode $r_0 = i \in \mathcal{D}$, there exists an $\tilde{M} > 0$ satisfying

$$\lim_{t_f\to\infty} \mathbb{E}\left[\int_0^{t_f} x^T(t,\phi,r_0)x(t,\phi,r_0)dt \,\Big|\, \phi,\ r_0 = i\right] \leq x^T(0)\tilde{M}x(0). \tag{4}$$

Lemma 1: [14] Let $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{q\times n}$ and $\Theta \in \mathbb{R}^{p\times q}$ satisfying $\Theta^T(r_t, t)\Theta(r_t, t) \leq I_q$ be given matrices. Then, for any matrix $P = P^T > 0$, there exist positive scalars $\varepsilon > 0$ and $\lambda > 0$ such that

$$(A + D\Theta E)^T P(A + D\Theta E) \leq A^T PA + \varepsilon^{-1} A^T PDD^T PA + (\varepsilon + \lambda)E^T E, \tag{5a}$$

$$D^T PD \leq \lambda I_p. \tag{5b}$$

The following result has been established as the extended stochastic version of the bounded real lemma for the existing result in [17].

Theorem 1: Let γ denote the required disturbance attenuation level. Consider a set of symmetric positive semidefinite matrices $W(i) \geq 0$, $U > 0$ and positive scalars $\varepsilon(i)$ and $\mu(i)$, such that the following CCSMIs holds for every $i \in \mathcal{D}$:

$$\Lambda(\boldsymbol{W}, \mu(i), \varepsilon(i), \lambda(i), i) < 0, \tag{6a}$$

$$D_p^T(i)W(i)D_p(i) \leq \lambda(i)I_{n_b}, \tag{6b}$$

where $i = 1, ..., s$, $\Lambda(\boldsymbol{W}, \mu(i), \varepsilon(i), \lambda(i), i) := \begin{bmatrix} \Phi^{11}(i) & \Phi^{12}(i) \\ \Phi^{12T}(i) & -U \end{bmatrix}$, $\Phi^{11}(i) := W(i)A(i) + A^T(i)W(i) + \mu^{-1}(i)W(i)D(i)D^T(i)W(i) + \mu(i)E_a^T(i)E_a(i) + H^T(i)H(i) + U + \sum_{j=1}^{s} \pi_{ij}W(j) + A_p^T(i)W(i)A_p(i) + \varepsilon^{-1}(i)A_p^T(i)W(i)D_p(i)D_p^T(i)W(i)A_p(i) + (\varepsilon(i) + \lambda(i))E_{pa}^T(i)E_{pa}(i) + \gamma^{-2}W(i)B_v(i)B_v^T(i)W(i)$, $\Phi^{12}(i) := W(i)A_h(i)$.

Then, we have the following results:

i) The UMJDSS (2) is stochastically stable internally with $v(t) \equiv 0$.

ii) The following inequality holds:

$$\|z\|_2^2 < \gamma^2\|v\|_2^2 + \mathcal{E}(W(i), U), \tag{7}$$

where $\|z\|_2^2 := \mathbb{E}\left[\int_0^\infty \|z(t)\|^2 dt \,\Big|\, r_0 = i\right]$, $\|v\|_2^2 := \mathbb{E}\left[\int_0^\infty \|v(t)\|^2 dt \,\Big|\, r_0 = i\right]$, $\mathcal{E}(W(i), U) := x^T(0)W(i)x(0) + \int_{-h}^{0} \phi^T(s)U\phi(s)ds$.

iii) The worst-case disturbance is given by

$$v^*(t) = F_\gamma^*(r_t)x(t) = \gamma^{-2}B_v^T(r_t)W(r_t)x(t). \tag{8}$$

Proof: First, the following Lyapunov function candidate is defined by $W(i) = W^T(i) > 0$,

$$V_v(x, t, i) := x^T(t)W(i)x(t) + \int_{t-h}^{t} x^T(s)Ux(s)ds. \tag{9}$$

By using Itô formula with the infinitesimal generator $\mathscr{L}$ [2], [17] and inequality (5), the stochastic differential can be obtained as

$$\mathscr{L}V_v(x, t, i) + \|z(t)\|^2 - \gamma^2\|v(t)\|^2 \leq \xi^T(t)\Lambda(\boldsymbol{W}, \mu(i), \varepsilon(i), \lambda(i), i)\xi(t) - \gamma^2\|v(t) - v^*(t)\|^2 \tag{10}$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-h) \end{bmatrix}$.

Hence, if $v(t) = v^*(t)$, we have

$$\xi^T(t)\Lambda(\boldsymbol{W}, \mu(i), \varepsilon(i), \lambda(i), i)\xi(t) < 0. \tag{11}$$

On the other hand, we have

$$\mathscr{L}V_v(x, t, i) + \|z(t)\|^2 - \gamma^2\|v(t)\|^2 < 0. \tag{12}$$

Thus, if $v(t) \equiv 0$, $\mathscr{L}V_v(x, t, i) < 0$ holds. Therefore, by using the technique similar to the proof in [16], [17], the following inequality holds:

$$\begin{aligned} &\lim_{t_f \to \infty} \mathbb{E}\left[\int_0^{t_f} x^T(t, \phi, r_0)x(t, \phi, r_0)dt\middle|\phi,\ r_0 = i\right] \\ &\quad \le x^T(0)\tilde{W}x(0), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tilde{W} &= \max_{i \in \mathcal{D}}\left\{\frac{\lambda_{\max}[W(i)] + \sigma^2 h\lambda_{\max}[U]}{\alpha\lambda_{\min}[W(i)]}\right\} I_n, \\ \alpha &:= \min_{i \in \mathcal{D}}\left\{\frac{\lambda_{\min}[-\Lambda(\boldsymbol{W}, \mu(i), \varepsilon(i), \lambda(i), i)]}{\lambda_{\max}[W(i)] + \sigma^2 h\lambda_{\max}[U]}\right\}. \end{aligned}$$

Hence, the UMJDSS in (2) is stochastically stable. Furthermore, using Dynkin's formula, we obtain

$$\begin{aligned} J_v(x(0), t_f, r_0) &\le -\mathbb{E}\left[\int_0^{t_f} \mathscr{L}V_v(x, t, i)dt\middle|r_0 = i\right] \\ &< \mathcal{E}(W(i), U), \end{aligned} \tag{14}$$

where $J_v(x(0), t_f, r_0) := \|z\|_2^2 - \gamma^2\|v\|_2^2$.

Moreover, considering that the UMJDSS is stochastically stable when t_f approaches infinity, inequality (7) holds. ■

The following corollary can be established.

Corollary 1: Define the corresponding cost function for UMJDSS (2) with $v(t) \equiv 0$ as follows:

$$\tilde{J} := \mathbb{E}\left[\int_0^\infty x^T(t, \phi, r_0)Q(r_t)x(t, \phi, r_0)dt\middle|\phi,\ r_0 = i\right], \tag{15}$$

where $Q(r_t) = Q^T(r_t) > 0$. Consider a set of symmetric positive semidefinite matrices $P(i) \ge 0$, $V > 0$ and positive scalars $\epsilon(i)$ and $\nu(i)$, such that the following CCSMIs holds:

$$\Gamma(\boldsymbol{P}, V, \nu(i), \epsilon(i), \kappa(i), i) < 0, \tag{16a}$$

$$D_p^T(i)P(i)D_p(i) \le \kappa(i)I_{n_b}, \tag{16b}$$

where $i = 1, ..., s$, $\Gamma(\boldsymbol{P}, V, \nu(i), \epsilon(i), \kappa(i), i) := \begin{bmatrix} \Psi^{11}(i) & \Psi^{12}(i) \\ \Psi^{12T}(i) & -V \end{bmatrix}$, $\Psi^{11}(i) := P(i)A(i) + A^T(i)P(i) + \nu^{-1}(i)P(i)D(i)D^T(i)P(i) + \nu(i)E_a^T(i)E_a(i) + Q(i) + V + \sum_{j=1}^s \pi_{ij}P(j) + A_p^T(i)P(i)A_p(i) + \epsilon^{-1}(i)A_p^T(i)P(i)D_p(i)D_p^T(i)P(i)A_p(i) + (\epsilon(i) + \kappa(i))E_{pa}^T(i)E_{pa}(i)$, $\Psi^{12}(i) := P(i)A_h(i)$.

Then,we have the following inequality

$$\tilde{J} < x^T(0)P(i)x(0) + \int_{-h}^0 \phi^T(s)V\phi(s)ds. \tag{17}$$

Proof: Let us define the Lyapunov function candidate of $V_x(x, t, i) := x^T(t)P(i)x(t) + \int_{t-h}^t x^T(s)Vx(s)ds$. Since this proof can be done by tracing the previous result in proof of Theorem 1, it is omitted. ■

III. PROBLEM FORMULATION

Consider the following UMJDSS:

$$\begin{aligned} dx(t) &= \left[A(r_t, t)x(t) + \sum_{m=1}^N B_m(r_t)u_m(t)\right. \\ &\qquad \left. + B_v(r_t)v(t)\right]dt + A_p(r_t, t)x(t)dw(t), \end{aligned} \tag{18a}$$

$$x(t) = \phi(t),\ t \in [-h,\ 0], \tag{18b}$$

$$z(t) = \begin{bmatrix} H(r_t)x(t) \\ G_1(r_t)u_1(t) \\ \vdots \\ G_N(r_t)u_N(t) \end{bmatrix}, \tag{18c}$$

$$y_k(t) = C_k(r_t)x(t) + C_{hk}(r_t)x(t-h), \tag{18d}$$

where $u_k(t) \in \mathbb{R}^{m_k}$, $k = 1, ..., N$, denote the k-th control input, and $y_k(t) \in \mathbb{R}^{r_k}$, $k = 1, ..., N$ denote the k-th output. Without loss of generality, it is assumed that $G_k^T(r_t)G_k(r_t) = I_{m_k}$. Furthermore, in order to eliminate the dependence of the cost performance on $x(0)$, it is assumed that $\mathbb{E}[x(0)] = 0$, $\mathbb{E}[x(0)x^T(0)] = I_n$.

The robust Nash strategy for multiple decision makers is investigated as an extension of the existing result reported in [14], in the sense that the delay is considered. Here, the problem under consideration is formulated as follows.

For a given $\gamma > 0$, find an SOF *strategy set $(u_1^*, ..., u_N^*)$ and a worst case disturbance v^**

$$\begin{aligned} u_k(t) &= u_k^*(t) = F_k^*(r_t)y_k(t) \\ &= F_k^*(r_t)C_k(r_t)x(t) + F_k^*(r_t)C_{hk}(r_t)x(t-h), \end{aligned} \tag{19a}$$

$$v(t) = v^*(t) = F_\gamma^*(r_t)x(t) \tag{19b}$$

such that

(i) $u_k(t) = u_k^*(t)$, $k = 0,\ 1,\ \dots,N$, *make* UMJDSS (18) *in the sense of stable internally and the following inequality holds:*

$$\|z\|_2^2 < \gamma^2\|v\|_2^2 + \mathcal{G}(\tilde{W}(i), \tilde{U}), \tag{20}$$

where $\mathcal{G}(\tilde{W}(i), \tilde{U}) := x^T(0)\tilde{W}(i)x(0) + \int_{-h}^0 \phi^T(s)\tilde{U}\phi(s)ds$.

(ii) *when $v(t) = v^*(t) = F_\gamma^*(r_t)x(t)$ is applied,*

$$\begin{aligned} &\tilde{J}_k(u_1^*, ..., u_{k-1}^*, u_k^*, u_{k+1}^*, ..., u_N^*, v^*) \\ &\quad \le \tilde{J}_k(u_1^*, ..., u_{k-1}^*, u_k, u_{k+1}^*, ..., u_N^*, v^*), \end{aligned} \tag{21}$$

where for $Q_k(r_t) = Q_k^T(r_t) > 0$ and $R_k(r_t) = R_k^T(r_t) > 0$,

$$\tilde{J}_k(u_1, ..., u_N, v, i) = \sup_{\Theta(r_t, t)} J_k(u_1, ...u_N, v, i), \tag{22a}$$

$$\begin{aligned} &J_k(u_1, ...u_N, v, i) \\ &= \mathbb{E}\left[\int_0^\infty x^T(t, \phi, r_0)\Big(Q_k(r_t) + C_k^T(r_t)F_k^T(r_t)R_k(r_t)\right. \\ &\qquad \left. \times F_k(r_t)C_k(r_t)\Big)x(t, \phi, r_0)dt\middle|\phi,\ r_0 = i\right]. \end{aligned} \tag{22b}$$

A. DISTURBANCE ATTENUATION CONDITION

The disturbance attenuation condition is derived in the following. Consider the closed-loop UMJLSS and the cost functions. For arbitrary $u_k(t) = F_k(r_t)y_k(t) = F_k(r_t)C_k(r_t)x(t)$, $k = 1, \ldots, N$, the closed-loop UMJDSS is established as

$$dx(t) = \left[\hat{A}(r_t,t)x(t) + \bar{A}_h(r_t)x(t-h) + B_v(r_t)v(t)\right]dt \\ +A_p(r_t,t)x(t)dw(t), \tag{23a}$$

$$z(t) = \begin{bmatrix} H(r_t)x(t) \\ G_1(r_t)F_1(r_t)C_1(r_t) \\ \vdots \\ G_N(r_t)F_N(r_t)C_N(r_t) \end{bmatrix} x(t), \tag{23b}$$

where $\hat{A}(i,t) := \bar{A}(i) + D(i)\Theta(i,t)E_a(i)$. $\bar{A}(i) := A(i) + \sum_{m=1}^{N} B_m(i)F_m(i)C_m(i)$, $\bar{A}_h(i) := \sum_{m=1}^{N} B_m(i)F_m(i)C_{hm}(i)$.

Thus, we have the following CCSMIs, using Theorem 1:

$$\tilde{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) < 0, \tag{24a}$$

$$D_p^T(i)\tilde{W}(i)D_p(i) \leq \tilde{\lambda}(i)I_{n_b}, \tag{24b}$$

where $i = 1, \ldots, s$, $\tilde{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) := \begin{bmatrix} \tilde{\Phi}^{11}(i) & \tilde{\Phi}^{12}(i) \\ \tilde{\Phi}^{12T}(i) & -\tilde{U} \end{bmatrix}$, $\tilde{\Phi}^{11}(i) := \tilde{W}(i)\bar{A}(i) + \bar{A}^T(i)\tilde{W}(i) + \tilde{\mu}^{-1}(i)\tilde{W}(i)D(i)D^T(i)\tilde{W}(i) + \tilde{\mu}(i)E_a^T(i)E_a(i) + H^T(i)H(i) + \sum_{m=1}^{N} C^T(i)\tilde{F}^T(i)\tilde{F}(i)C(i) + \tilde{U} + \sum_{j=1}^{s}\pi_{ij}\tilde{W}(j) + A_p^T(i)\tilde{W}(i)A_p(i) + \tilde{\varepsilon}^{-1}(i)A_p^T(i)\tilde{W}(i)D_p(i)D_p^T(i)\tilde{W}(i)A_p(i) + (\tilde{\varepsilon}(i) + \tilde{\lambda}(i))E_{pa}^T(i)E_{pa}(i) + \gamma^{-2}\tilde{W}(i)B_v(i)B_v^T(i)\tilde{W}(i)$, $\tilde{\Phi}^{12}(i) := \tilde{W}(i)\bar{A}_h(i)$.

Furthermore, the worst-case disturbance is given by

$$v^*(t) = \gamma^{-2}B_v^T(r_t)\tilde{W}(r_t)x(t). \tag{25}$$

B. NASH EQUILIBRIUM CONDITION

Next, the Nash equilibrium condition is established. Using Corollary 1, the following CCSMIs can be obtained:

$$\tilde{\Gamma}_k(\boldsymbol{P}_k, \tilde{V}_k, \tilde{\nu}_k(i), \tilde{\epsilon}_k(i), \tilde{\kappa}_k(i), i) < 0, \tag{26a}$$

$$D_p^T(i)P_k(i)D_p(i) \leq \tilde{\kappa}_k(i)I_{n_b}, \tag{26b}$$

where $i = 1, \ldots, s$, $\tilde{\Gamma}_k(\boldsymbol{P}_k, \tilde{V}_k, \tilde{\nu}_k(i), \tilde{\epsilon}_k(i), \tilde{\kappa}_k(i), i) := \begin{bmatrix} \tilde{\Psi}^{11}(i) & \tilde{\Psi}^{12}(i) \\ \tilde{\Psi}^{12T}(i) & -\tilde{V}_k \end{bmatrix}$, $\tilde{\Psi}^{11}(i) := P_k(i)\bar{A}_\gamma(i) + \bar{A}_\gamma^T(i)P_k(i) + \tilde{\nu}_k^{-1}(i)P_k(i)D(i)D^T(i)P_k(i) + \tilde{\nu}_k(i)E_a^T(i)E_a(i) + Q_k(i) + C_k^T(i)F_k^T(i)R_k(i)F_k(i)C_k(i) + \tilde{V}_k + \sum_{j=1}^{s}\pi_{ij}P_k(j) + A_p^T(i)P_k(i)A_p(i) + \tilde{\epsilon}_k^{-1}(i)A_p^T(i)P_k(i)D_p(i)D_p^T(i)P_k(i)A_p(i) + (\tilde{\epsilon}_k(i) + \tilde{\kappa}_k(i))E_{pa}^T(i)E_{pa}(i)$, $\tilde{\Psi}^{12}(i) := P_k(i)\bar{A}_h(i)$, $\bar{A}_\gamma(i) := \bar{A}(i) + B_v(i)F_\gamma(i)$.

Consequently, the robust SOF Nash strategy for UMJDSS can be obtained by solving the following upper bound minimization problem of cost function (22b):

$$\min_{u_k} \bar{J}_k(u_1^*, \ldots, u_{k-1}^*, u_k, u_{k+1}^*, \ldots, u_N^*, v^*, i) \\ = \min_{\boldsymbol{\Sigma}_0} \mathbf{Tr}[P_k(i) + LL^T\tilde{V}_k], \tag{27}$$

s.t. $\boldsymbol{\Sigma}_0 := (\boldsymbol{P}_k, \boldsymbol{F}_k, \tilde{V}_k, \tilde{\boldsymbol{\nu}}_{\boldsymbol{k}}, \tilde{\boldsymbol{\epsilon}}_{\boldsymbol{k}}, \bar{\boldsymbol{\kappa}}_k)$ satisfies (26).

where u_m^*, $m = 1, \ldots, k-1, k+1, \ldots, N$, are the fixed strategy and $LL^T := \int_{-h}^{0} \phi(s)\phi^T(s)ds$.

However, it is difficult to solve this optimization problem, because this problem is formulated within the general bilinear matrix inequalities (BMIs). Therefore, a new algorithm for solving such BMIs is proposed by applying the iterative technique based on linear matrix inequalities (LMIs) and the coupled matrix equations.

First, inequality constraints (26a) can be changed to the following equations, by using the Schur's method and equivalence between the equations and LMIs [18]:

$$\Delta_k(\boldsymbol{P}_k, \tilde{\nu}_k(i), \tilde{\epsilon}_k(i), \tilde{\kappa}_k(i), i) \\ = \tilde{\Psi}^{11}(i) + \tilde{\Psi}^{12}(i)\tilde{V}_k^{-1}\tilde{\Psi}^{12T}(i) = 0. \tag{28}$$

The existence conditions can be established by using the method of Lagrange multipliers. The Lagrangian, $\boldsymbol{L}_k(i)$, is defined as

$$\boldsymbol{L}_k(i) = \mathbf{Tr}\ [P_k(i)] + \mathbf{Tr}[LL^T\tilde{V}_k] + \sum_{j=1}^{s}\mathbf{Tr}\ [S_k(j) \\ \times\Delta_k(\boldsymbol{P}_k, \tilde{\nu}_k(j), \tilde{\epsilon}_k(j), \tilde{\kappa}_k(j), j)], \tag{29}$$

where $S_k(j)$ is the symmetric matrix of the Lagrange multiplier, and we set $r_0 = i$.

In this case, we have the following cross coupled stochastic matrix equations (CCSMEs):

$$\frac{\partial \boldsymbol{L}_k(i)}{\partial P_k(i)} = \Delta_k^1,\ \frac{1}{2}\cdot\frac{\partial \boldsymbol{L}_k(i)}{\partial F_k(i)} = \Delta_k^2, \tag{30}$$

where $\Delta_k^1 = I_n + S_k(i)\bar{A}_\gamma^T(i) + \bar{A}_\gamma(i)S_k(i) + \tilde{\nu}_k^{-1}(i)\left[S_k(i)P_k(i)D(i)D^T(i) + D(i)D^T(i)P_k(i)S_k(i)\right] + \sum_{j=1}^{s}\pi_{ji}S_k(j) + A_p(i)S_k(i)A_p^T(i) + \tilde{\epsilon}_k^{-1}(i)[A_p(i)S_k(i)A_p^T(i)P_k(i)D_p(i)D_p^T(i) + D_p(i)D_p^T(i)P_k(i)A_p(i)S_k(i)A_p^T(i)] + \left[S_k(i)P_k(i)\bar{A}_h^T(i)\tilde{V}_k^{-1}\bar{A}_h(i) + \bar{A}_h^T(i)\tilde{V}_k^{-1}\bar{A}_h(i)P_k(i)S_k(i)\right]$, $\Delta_k^2 = \sum_{m=1}^{N} B_k^T(i)P_k(i)S_k(i)P_k(i)B_m(i)F_m(i)C_{hm}(i) \times \tilde{V}_k^{-1}C_{hk}^T(i) + R_k(i)F_k(i)C_k(i)S_k(i)C_k^T(i) + B_k^T(i)P_k(i)S_k(i)C_k^T(i)$.

Hence, the following heuristic algorithm for solving the optimization problem (27) is proposed.

Step 1. *Set the initial values: choose $F_k^{(0)}(i)$, $k = 1, \ldots, N$, and $F_\gamma^{(0)}(i)$, $i = 1, \ldots, s$, such that closed-loop* UMJDSS (18a) *is stochastically stable; choose an appropriate ρ value for $\tilde{W}^{(0)}(i) = \rho I_n$;*

Step 2-1. *Solve the following optimization problem for $P_k^{(n+1)}(i)$ and $V_k^{(n+1)}$ for variable $\boldsymbol{\Sigma}_1$:*

$$\min_{\boldsymbol{\Sigma}_1} \mathbf{Tr}\left[\sum_{j=1}^{s} P_k^{(n+1)}(j) + LL^T\tilde{V}_k^{(n+1)}\right], \tag{31a}$$

$\boldsymbol{\Sigma}_1 := (\boldsymbol{P}_k^{(n+1)}, \tilde{V}_k^{(n+1)}, \tilde{\boldsymbol{\nu}}_k^{(n+1)}, \tilde{\boldsymbol{\epsilon}}_k^{(n+1)}, \tilde{\boldsymbol{\kappa}}_k^{(n+1)})$,

s.t. $\boldsymbol{\Sigma}_1$ satisfies (31b) and (31c),

$\hat{\Gamma}_k(\boldsymbol{P}_k, \tilde{V}_k, \tilde{\nu}_k(i), \tilde{\epsilon}_k(i), \tilde{\kappa}_k(i), i)$

$$:= \begin{bmatrix} \mathbf{\Psi}^{11} & \tilde{\Psi}^{12}(i) & \tilde{\Psi}^{13}(i) & \tilde{\Psi}^{14}(i) \\ \tilde{\Psi}^{12T}(i) & -\tilde{V}_k & 0 & 0 \\ \tilde{\Psi}^{13T}(i) & 0 & -\tilde{\nu}_k(i)I_{n_p} & 0 \\ \tilde{\Psi}^{14T}(i) & 0 & 0 & -\tilde{\epsilon}_k(i)I_{n_p} \end{bmatrix} < 0, \tag{31b}$$

$$D_p^T(i)P_k(i)D_p(i) \le \tilde{\kappa}_k(i)I_{n_b}, \tag{31c}$$

where $i = 1, ..., s$, $\mathbf{\Psi}^{11} := P_k(i)\bar{A}_\gamma(i) + \bar{A}_\gamma^T(i)P_k(i) + \tilde{\nu}_k(i)E_a^T(i)E_a(i) + Q_k(i) + C_k^T(i)F_k^T(i)R_k(i)F_k(i)C_k(i) + \tilde{V}_+ \sum_{j=1}^s \pi_{ij}P_k(j) + A_p^T(i)P_k(i)A_p(i) + (\tilde{\epsilon}_k(i) + \tilde{\kappa}_k(i))E_{pa}^T(i)E_{pa}(i)$, $\tilde{\Psi}^{13}(i) := P_k(i)D(i)$, $\tilde{\Psi}^{14}(i) := A_p^T(i)P_k(i)D_p(i)$.

Step 2-2. *Solve the following* CCSMEs *for* $\boldsymbol{S}_k^{(n+1)}$*:*

$$\Delta_k^1(\boldsymbol{S}_k^{(n+1)}, \tilde{W}^{(n)}(i), P_k^{(n+1)}(i), \boldsymbol{F}_1^{(n)}, ..., \boldsymbol{F}_N^{(n)}, \tilde{\nu}_k^{(n+1)}(i), \tilde{\epsilon}_k^{(n+1)}(i), \tilde{\kappa}_k^{(n+1)}(i), i) = 0. \tag{32}$$

Step 2-3. *Solve the following* CCSMEs *for* $F_k^{(n+1)}(i)$*,* $k = 1, ..., N$*:*

$$\Delta_k^2(\boldsymbol{S}_k^{(n+1)}, \tilde{W}^{(n)}(i), P_k^{(n+1)}(i), \boldsymbol{F}_1^{(n+1)}, ..., \boldsymbol{F}_N^{(n+1)}, \tilde{\nu}_k^{(n+1)}(i), \tilde{\epsilon}_k^{(n+1)}(i), \tilde{\kappa}_k^{(n+1)}(i), i) = 0. \tag{33}$$

Step 3. *Solve the following optimization problem for* $\tilde{W}^{(n+1)}(i)$ *for the variables* $\mathbf{\Sigma}_2$*:*

$$\min_{\mathbf{\Sigma}_2} \sum_{j=1}^s \mathbf{Tr}[\tilde{W}^{(n+1)}(j)], \tag{34a}$$

$$\mathbf{\Sigma}_2 := (\tilde{\boldsymbol{W}}^{(n+1)}, \tilde{\boldsymbol{\mu}}^{(n+1)}, \tilde{\boldsymbol{\varepsilon}}^{(n+1)}, \tilde{\boldsymbol{\lambda}}^{(n+1)}),$$

s.t. $\mathbf{\Sigma}_2$ satisfies (34b) and (34c),

$$\hat{\Lambda}(\tilde{\boldsymbol{W}}, \tilde{\mu}(i), \tilde{\varepsilon}(i), \tilde{\lambda}(i), i) := \begin{bmatrix} \mathbf{\Phi}^{11} & \tilde{\Phi}^{12}(i) & \tilde{\Phi}^{13}(i) & \tilde{\Phi}^{14}(i) & \tilde{\Phi}^{15}(i) \\ \tilde{\Phi}^{12T}(i) & -\tilde{U} & 0 & 0 & 0 \\ \tilde{\Phi}^{13T}(i) & 0 & -\tilde{\mu}(i)I_{n_p} & 0 & 0 \\ \tilde{\Phi}^{14T}(i) & 0 & 0 & -\tilde{\varepsilon}(i)I_{n_p} & 0 \\ \tilde{\Phi}^{15T}(i) & 0 & 0 & 0 & -\gamma^2 I_{m_v} \end{bmatrix} < 0, \tag{34b}$$

$$D_p^T(i)\tilde{W}(i)D_p(i) \le \tilde{\lambda}(i)I_{n_b}, \tag{34c}$$

where $i = 1, ..., s$, $\mathbf{\Phi}^{11} := \tilde{W}(i)\bar{A}(i) + \bar{A}^T(i)\tilde{W}(i) + \tilde{\mu}(i)E_a^T(i)E_a(i) + H^T(i)H(i) + \sum_{m=1}^N C^T(i)\tilde{F}^T(i)\tilde{F}(i)C(i) + \tilde{U} + \sum_{j=1}^s \pi_{ij}\tilde{W}(j) + A_p^T(i)\tilde{W}(i)A_p(i) + (\tilde{\varepsilon}(i) + \tilde{\lambda}(i))E_{pa}^T(i)E_{pa}(i)$, $\tilde{\Phi}^{12}(i) := \tilde{W}(i)\bar{A}_h(i)$, $\tilde{\Phi}^{13}(i) := \tilde{W}(i)D(i)$, $\tilde{\Phi}^{14}(i) := A_p^T(i)\tilde{W}(i)D_p(i)$, $\tilde{\Phi}^{15}(i) := \tilde{W}(i)B_v(i)$.

Step 4. *Set*

$$\begin{aligned} &\left[\mathrm{vec}[\boldsymbol{P}_1^{(n+1)}] \cdots \mathrm{vec}[\boldsymbol{P}_N^{(n+1)}]\ \mathrm{vec}[\boldsymbol{W}^{(n+1)}]\right] \\ &\leftarrow \theta^{(n)} \left[\mathrm{vec}[\boldsymbol{P}_1^{(n+1)}] \cdots \mathrm{vec}[\boldsymbol{P}_N^{(n+1)}]\ \mathrm{vec}[\boldsymbol{W}^{(n+1)}]\right] \\ &\quad + (1-\theta^{(n)}) \left[\mathrm{vec}[\boldsymbol{P}_1^{(n)}] \cdots \mathrm{vec}[\boldsymbol{P}_N^{(n)}]\ \mathrm{vec}[\boldsymbol{W}^{(n)}]\right], \end{aligned} \tag{35}$$

where $\theta^{(n)} \in (0, 1]$ *is chosen at each iteration to ensure that* $\mathscr{J}^{(n)} > \mathscr{J}^{(n+1)}$ *with* $\mathscr{J}^{(n)} = \sum_{m=1}^N \sum_{j=1}^s \mathbf{Tr}[P_m^{(n)}(j)] + \sum_{j=1}^s \mathbf{Tr}[W^{(n)}(j)]$*;*

Step 5. *If the iterative algorithm consisting of Steps 2 to 4 converges, we have obtained the iterative solutions for the CCSMIs* (24) *and* (26)*; otherwise, if the number of iterations reaches a preset threshold, declare that there is no strategy set. Stop.*

The proposed heuristic algorithm is called KM iteration [15] and expected to obtain a robust Nash SOF strategy, because the algorithm always generates a non-increasing sequence of the cost. That is, the following property is guaranteed:

$$\mathscr{J}^{(0)} > \cdots > \mathscr{J}^{(n)} > \mathscr{J}^{(n+1)} > \cdots > 0. \tag{36}$$

Theorem 2: The algorithm based on the KM iteration achieves the convergence if there exists $\theta^{(n)} \in (0, 1]$ such that for all $n \in \mathbb{N}$, $\mathscr{J}^{(n)} > \mathscr{J}^{(n+1)}$. Furthermore, a converged solution set satisfies (28), (30) and

$$\tilde{\Phi}^{11}(i) + \tilde{\Phi}^{12}(i)\tilde{U}^{-1}\tilde{\Phi}^{12T}(i) = 0. \tag{37}$$

IV. PRACTICAL EXAMPLE

Consider the linearized TCP/AQM (transmission control protocol/active queue management) fluid-flow model with three bottleneck routers [1]. It should be noted that the probability of a packet being marked or dropped can be considered the control input. The marking probability can be approximated by $\delta p_k(t-h)$ in [1] given that $\delta p_k(t-h) = F_k[\delta q_k(t) + \delta q_k(t-h)]$, $k = 1,\ 2$ are very small, where F_k is the SOF gain to be determined. For this linearized model, it is supposed that in the case of wireless networks, Markov jump system is considered by taking into account changes in link capacity and system noise is expressed by a Wiener process. Furthermore, since the delay h is sufficient small, it is assumed that $\delta\omega_\ell(t) + \delta\omega_\ell(t-h) = 2\delta\omega_\ell(t) + 0.01 \times \Theta(r_t, t)$ and $\delta q_k(t) - \delta q_k(t-h) = 0.01 \times \Theta(r_t, t)$ with $|\Theta(r_t, t)| \le 1$ for the deterministic uncertainty.

Finally, the state and the coefficient matrices in UMJDSS (18) with two modes as $s = 2$ are given below:

$$x(t) = \begin{bmatrix} \delta\omega_1(t) & \delta\omega_2(t) & \delta\omega_3(t) & \delta q_1(t) & \delta q_2(t) \end{bmatrix}^T,$$

$$\begin{aligned} u_k(t) &= F_k(r_t)y_k(r_t) = F_k(r_t)[\delta q_k(t) + \delta q_k(t-h)] \\ &= F_k(r_t)[C_k(r_t)x(t) + C_{hk}(r_t)x(t-h)],\ k = 1,\ 2, \end{aligned}$$

$$A(r_t, t) = \begin{bmatrix} A_{11}(r_t) & 0 \\ A_{21}(r_t) & A_{22}(r_t) \end{bmatrix},\ A_p(r_t, t) = 0.05A(r_t, t),$$

$$A_{11}(i) = -\mathbf{diag}\left(\frac{2}{h\omega_1^*(i)} \quad \frac{2}{h\omega_2^*(i)} \quad \frac{2}{h\omega_3^*(i)} \right),$$

$$A_{21}(i) = -\begin{bmatrix} \frac{N_1(i)}{h} & \frac{N_2(i)}{h} & 0 \\ \frac{N_1(i)}{h} & 0 & \frac{N_3(i)}{h} \end{bmatrix},$$

$$A_{22}(i) = -\mathbf{diag}\left(\frac{1}{h} \quad \frac{1}{h} \right),$$

$$B_1(i) = \begin{bmatrix} -\frac{hc_1^2(i)}{2N_1^2(i)} & -\frac{hc_2^2(i)}{2N_2^2(i)} & 0 & 0 & 0 \end{bmatrix}^T,$$

$$B_2(i) = \begin{bmatrix} -\frac{hc_1^2(i)}{2N_1^2(i)} & 0 & -\frac{hc_3^2(i)}{2N_3^2(i)} & 0 & 0 \end{bmatrix}^T,$$

$$B_v(i) = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.5 & 0.5 \end{bmatrix}^T,$$

$$D(i) = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0 & 0 \end{bmatrix}^T,\ D_p(i) = 0.1D(i),$$

$$C_1(i) = C_{h1}(i) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$C_2(i) = C_{h2}(i) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_a(i) = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0 & 0 \end{bmatrix},\ E_{pa}(i) = 0.1E_a(i),$$

$$G_1(i) = G_2(i) = 1,\ Q_1(i) = I_5,\ Q_2(i) = 2I_5,\ R_k(i) = 1.$$

As in [1], we set $N_1(1) = N_2(1) = 30$, $N_1(2) = N_2(2) = 40$ [TCP sessions], $\omega_1^*(1) = 80$, $\omega_2^*(1) = 40$, $\omega_3^*(1) = 50$, $\omega_1^*(2) = 70$, $\omega_2^*(2) = 50$, $\omega_3^*(2) = 60$ [packets]. $c_\ell(1) = 0.1$ [Mbps/sec], $c_\ell(2) = 0.2$ [Mbps/sec], $\ell = 1, 2, 3$, $h = 0.02$[s].

The value of disturbance attenuation level γ related to the H_∞ constraint is set to 5. We use the proposed iterative algorithm with KM iteration to obtain the converged solution and the strategy set. The initial gains are set as $F_k^{(0)}(i) = \left[\ -\rho\ \right]$, $k = 1,\ 2$, $F_\gamma^{(0)}(i) = \gamma^{-2} B_v^T(i)(\rho I_5)$, $\rho = 2$ for $i = 1,\ 2$. The initial condition has been chosen by trial and error, such that the closed loop system is stable. The algorithm converges after 661 iterations with an accuracy of 10^{-8} and $\theta^{(n)} = 0.5$. The robust Nash strategy set (19a) is given by

$$\begin{aligned}
&F_1^*(1) = \left[-1.9498\text{e}{-7}\right],\ F_2^*(1) = \left[-7.5692\text{e}{-8}\right],\\
&F_1^*(2) = \left[-2.2757\text{e}{-7}\right],\ F_2^*(2) = \left[-8.8672\text{e}{-8}\right],\\
&F_\gamma(1) = [\ 3.9636\text{e}{-4}\quad 2.8270\text{e}{-4}\quad 3.1999\text{e}{-4}\\
&\qquad\qquad 1.2580\text{e}{-15}\quad 1.5476\text{e}{-15}\ ],\\
&F_\gamma(2) = [\ 3.9597\text{e}{-4}\quad 3.2514\text{e}{-4}\quad 3.5884\text{e}{-4}\\
&\qquad\qquad 8.6294\text{e}{-16}\quad 1.0534\text{e}{-15}\ ].
\end{aligned}$$

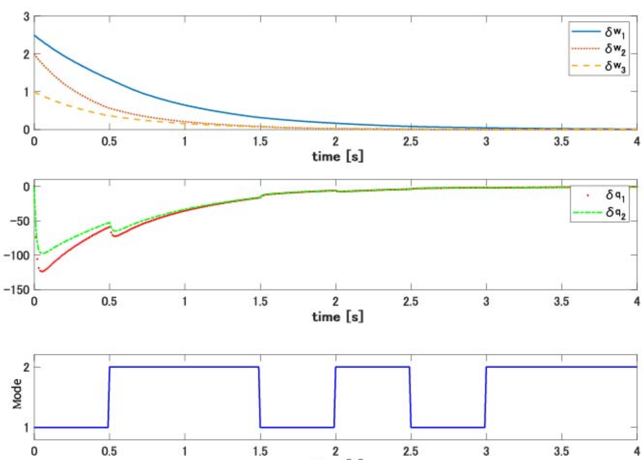

Fig. 1. State and mode trajectories of the TCP/AQM fluid-flow model.

Finally, Fig. 1 shows how the system states and mode vary with time, with the vector-valued initial continuous function defined by $\phi(t) = \left[\ 2.5\quad 2.0\quad 1\quad 0\quad 0\ \right]^T$, $-h \le t \le 0$. In this practical example, all the states converge to the equilibrium point.

V. Conclusion

In this paper, we have studied the robust SOF Nash strategies for the UMJDSSs. As the result, a strategy set can be obtained even if only local or partial information of the system states is available. Different from the recent studies in [14], the conditions of the extended bounded real lemma and the SOF Nash strategy set for a class of Markov jump delay stochastic system have been derived by means of the CCSMIs. The heuristic algorithm based on the LMI algorithm has been proposed. Furthermore, a novel convergence condition combined with the KM iteration is introduced to achieve robust convergence of the algorithms. Finally, the practical congestion control problem as a typical example of delay systems is solved to demonstrate the validity of the proposed algorithm, and effectiveness of combining the KM iteration for algorithm convergence.

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